A

GEOMETRY FOR BEGINNERS.

BY

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PREFACE.

THE formal method of teaching Geometry, as we find it in Euclid and in the common text-books, whatever may be its merits, is a very bad method for beginners. It makes the study of the subject unnecessarily dry, tedious, and difficult; and it ignores, in most cases, the great law of mental development, that clear perceptions and intuitions must precede the intelligent use of the faculties of comparison and reasoning. The usual consequence is that this method exercises the memory more than the intelligence, and utterly fails to impart any habits of thought that are useful in after-life.

In the present work, a method is employed which seems to the author much better suited to promote the natural growth of the mental powers. The method will speak for itself to those who will take the trouble to examine it. A marked feature of it consists in the numerous exercises which are to be worked by the learner. By this kind of work, if the exercises are well chosen and faithfully studied, the inventive faculty is called into play, and the habit of seizing quickly the true relations of things is acquired. And it is just here, — in its power to form this habit of mind, — that the value of the study of Geometry, as a preparation for the varied duties and labors of life, can hardly be over-estimated. Does not the possession or want of this habit constitute, in most instances, the chief intellectual difference between a man who succeeds in the world and a man who does not?

In Germany the true worth of Geometry as an educational means is better recognized than elsewhere; and the simpler parts of the subject,

treated much as in this work, have been introduced with the happiest results into the common schools. The author, while residing in Germany in 1877 and 1878, made himself familiar with their methods and text-books, and he has freely used the knowledge thus acquired in writing the present work.

As regards the subjects treated in the last four chapters, the author was not satisfied with what he found, even in the best German text-books, and he has planned and written these chapters according to his own ideas.

To make the study both more interesting and more useful, much attention has been given to the practical uses of Geometry; such, for example, as measuring inaccessible distances, and computing lengths, areas, and volumes.

In a few cases the method of Limits has been used. By this means rigor of proof has been secured; and when the method is presented in its simplest form (as on page 230), the author believes that it is quite within the comprehension of the average boy fifteen years old.

If answers to the exercises are generally desired they will soon be published.

The author desires to express his obligations to Prof. G. A. WENTWORTH, who has kindly read the proof-sheets, and furnished him with many valuable suggestions.

G. A. HILL.

CAMBRIDGE, June 1, 1880.

CONTENTS.

GEOMETRY FOR BEGINNERS.

CHAPTER I.

INTRODUCTION.

I.—Space.

§ **1**. We cannot define space. We know, however, that all things exist in space. I lay a book on the table: it has now a definite position in space on the table. The table is in the space occupied by the room; the room is in the space enclosed by the house; the house rests upon the earth; and the earth is always moving swiftly through space. Thus all things are in space; but what space is, where it begins, or where it ends, these are questions which no man can answer.

Space contains all things, and extends in all directions, without beginning and without end.

II.—The Cube.[1]

§ **2**. The Cube (*Fig. 1*) occupies a portion of space which is limited on all sides. A limited portion of space is called a BODY (see § **7**). Therefore the cube is a body.

[1] A model of the cube is supposed to rest on the table or other support, with one face turned towards the eye of the learner.

The cube is extended in space in *three* chief directions: from *right* to *left*, from *front* to *back*, from *above downwards*. We express this fact by saying that the cube has three DIMENSIONS, and we name them Length, Breadth, and Thickness (Height or Depth). Either one of the dimensions may be termed length or breadth or thickness, but the terms height and depth are applied only to the dimension from above downwards.

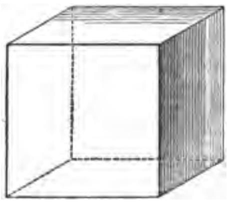

Fig. 1.

The three dimensions of the cube are equal in magnitude.

Exercises.—**1.** Here are several cubes. Have they all the same shape? the same size or magnitude? Are they all composed of the same *substance* or *material?* Why are they called bodies? Why are they called cubes?

2. Mention objects which are cubical in shape, or resemble a cube. (Lumps of sugar, pieces of soap, tea-chests, some boxes, etc.)

§ **3.** The cube is limited or bounded on all sides by a SURFACE. When we handle the cube it is its surface alone which we touch. If the cube is not transparent, it is its surface alone which we are able to see.

The entire surface consists of six parts or partial surfaces, called its Faces: the upper, lower, front, back, right, and left faces. Each face has *two* dimensions, length and breadth (height). Thus the upper face has length (extending from right to left) and breadth (extending from front to back); the front face has length and height, etc.

The faces of the cube are *flat* or *plane* surfaces.

Exercises.—**1.** Name the faces of the cube, and likewise the dimensions of each face.

2. Have the faces of the cube the same shape or form? the same size? How can you test whether they have the same size or not?

Answer.—Lay the cube on a piece of paper, mark around its base with a pencil, then apply to the surface thus marked out the other faces of the cube.

3. Give examples of plane surfaces both *like* and *unlike* the faces of the cube in form.

4. Do the faces of one cube have the same *shape* as those of another cube? the same *size?*

§ **4.** The faces of the cube are bounded by its Edges. These edges are LINES, and we see, (*a*) that each face is bounded by four lines or edges, (*b*) that each edge is also the place where two faces meet.

The edges of the cube have only *one* dimension: length.

The edges of the cube are *straight* lines.

Exercises.—1. How many edges has the cube in all?

2. If the cube has 6 faces, and each face has 4 edges, why has not the cube in all $6 \times 4 = 24$ edges?

3. Have the edges of the cube the same form? the same size (in other words, the same *length*)? Test whether they have the same length.

4. Of these cubes, which has the longest edges? Which the shortest?

§ **5.** The edges of the cube are limited by its Corners. These corners are POINTS, and we see, (*a*) that each edge is limited by two corners, (*b*) that each corner is the place where three edges meet.

The corners of the cube have *no* dimensions, neither length, breadth, nor thickness.

How many corners are there on the cube?

Exercises.

1. This body (*Fig. 2*, I.) is a three-sided (or triangular) Prism:—

(*a*) Point out and name its dimensions (height, length, and breadth).

NOTE.—The front edge gives the *height* of the prism. The other two dimensions are, (1) the distance from this edge to the opposite face, and (2) the upper or lower edge of this face; either of these may be taken as the *length*, and then the other will be the *breadth*.

(*b*) How many faces, edges, corners, are there? Which are surfaces, which lines, which points? (In prisms the upper and lower faces are called the *bases*, the other faces the *lateral* faces.)

(*c*) Point out and name the dimensions of each face.

(*d*) By how many edges (lines) is each face bounded?

(*e*) How many faces meet in each edge?

(*f*) How many edges meet at each corner?

(*g*) Have the faces the same shape (form)? the same size (magnitude)?

(*h*) Have the edges the same form? the same magnitude?

(*i*) Does *shape* or *size* or *material* determine whether a body is a three-sided Prism?

2. Examine in the same way the four-sided (square) Prism (*Fig. 2*, II.).

3. Examine in the same way the six-sided (hexagonal) Prism (*Fig. 2*, III.).

4. Examine in the same way the four-sided (rectangular) Pyramid (*Fig. 2*, IV.).

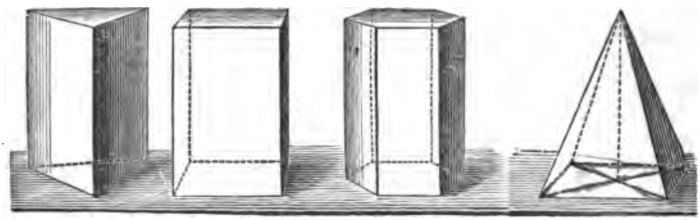

Fig. 2.

III.—The Cylinder.[1]

§ **6.** The Cylinder (*Fig. 3*) is also, like the cube, a limited portion of space: in other words, a body. Its three dimensions are usually called length, breadth, and height. The height (the straight dotted line in the figure) may have any value: so may the length and breadth, but they must be always equal to each other.

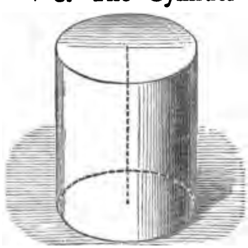

Fig. 3.

The cylinder is bounded by three surfaces: two plane surfaces called its Bases, and a *curved* surface. The bases of the cylinder are Circles.

1 Several models of right cylinders are supposed to be placed on the teacher's desk before the eyes of the pupils.

The cylinder has only two edges; these form the boundaries of the bases, and are called Circumferences. They are not straight lines, but *curved* lines.

The cylinder has no corners.

Exercises.—**1.** Here are several cylinders. How do they differ from one another? In what do they agree?

2. Give examples of objects which have the shape of cylinders (a pencil, stove-funnel, etc.).

3. Examine, as regards dimensions, surface, etc., the Cone (*Fig. 4*, I.).

4. Examine in like manner the Truncated[1] Cone (*Fig. 4*, II.).

5. Examine in like manner the Sphere (*Fig. 4*, III.).

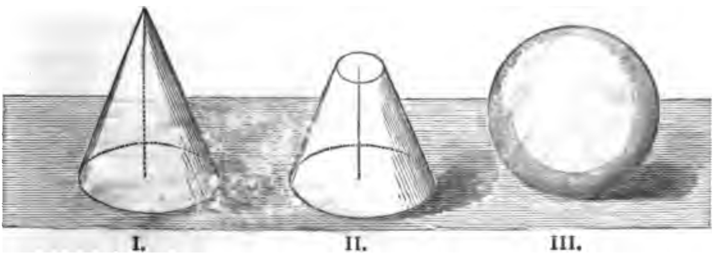

Fig. 4.

IV.—Bodies, Surfaces, Lines, and Points.

§ **7.** A body, in common language, is a limited portion of space *filled with matter*, which we can see, touch, handle, etc.; for example, a pencil, a book, the table, a tree, the earth, the sun. But in Geometry we pay no regard to the matter of which a body is composed; we study simply its *shape* and its *size:* in other words, we regard it simply as a limited portion of space.

Definition.—A GEOMETRICAL BODY, *or* SOLID, *is a limited portion of space.*

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[1] *Truncated* means *cut off.* The body is also called the *Frustum* of a cone.

A body has three dimensions, denoted by the terms length, breadth, and thickness (height and depth).

NOTE.—Either dimension may be called length, or breadth, or thickness. A dimension estimated from below upwards, is often called *height*, as the height of a monument; a dimension estimated from above downwards, is often called *depth*, as the depth of a well. *Width* is also used in the sense of *breadth*.

§ **8.** The limits or boundaries of a body are surfaces. Surfaces exist not only on the exterior of a body, they may also be thought of as existing in its interior. When we cut an apple into two parts and then hold the parts together, the common boundary between the two parts is a surface. But without actually cutting through a body we may imagine a surface extending in any direction completely through the body. This surface would divide the body into two parts and form the common boundary between them. Every body is divisible into parts, and the common boundary between two adjacent parts is a surface.

Definition.—A SURFACE *is the limit of a body, or common boundary between two adjacent parts of a body.*

A surface has two dimensions, length and breadth (height).

§ **9.** The limits of a surface are lines. Lines exist not only as the limits or boundaries of surfaces: we can also think of them as existing anywhere in a surface, where they form the common boundaries between adjacent parts of the surface.

Definition.—A LINE *is the limit of a surface, or the common boundary between two adjacent parts of a surface.*

A line has one dimension, length.

§ **10.** The limits or ends of a line are points. Points exist not only at the ends of lines, but also between the ends, where they form the common boundaries between adjacent parts of the line.

Definition.—A POINT *is the limit of a line, or common boundary between two adjacent parts of a line.*

A point has no dimensions.

§ **11.** Bodies, surfaces, lines, and points are four different kinds of things relating to space which we can think and reason about.

Definition.—*Bodies, surfaces, lines, and points are called* SPACE CONCEPTIONS.

The first three: bodies, surfaces, and lines, are *extended* in space, and hence possess *shape* (*form*) and *size* (*magnitude*).

Definition.—*With reference to form, bodies, surfaces, and lines are called* SPACE FIGURES; *with reference to magnitude, they are called* SPACE MAGNITUDES.

The point has no extension, therefore neither form nor magnitude.

§ **12.** We have seen that two faces of the cube, prism, etc., meet in an edge, that is to say, in a *line*. Whenever two surfaces meet they are said to *cut* or *intersect* each other; and the place of meeting is a line called their Intersection. This line lies at once in both surfaces, or is *common* to both surfaces.

Again, as we have seen, the edges of the cube, etc., meet at the corners; that is to say, in *points*. Whenever two lines meet, the place of meeting is a point common to both lines, and is called their intersection.

Definition.—*The place where two surfaces or two lines meet is called their* INTERSECTION.

The intersection of two surfaces is a line.

The intersection of two lines is a point.

§ **13.** Divide a body (for example, an apple) into *any number* of parts; each part is also a body, smaller of course than the whole body, because a part of a thing is less than the whole.

Grind a lump of salt into the finest particles you can, so fine that you cannot distinguish them by the eye. Under a microscope, each particle is seen to be what it really is,—a little lump of salt, having length, breadth, and thickness: in other words, a

body. In short, subdivide a body in thought as far as you please, the smallest conceivable part is also a body, occupying space, and having length, breadth, and thickness. We could never arrive at a part which was a surface, a line, or a point.

In like manner, every part of a surface, however small, is also a surface; and every part of a line is also a line.

The parts of a body are bodies, the parts of a surface are surfaces, and the parts of a line are lines.

§ **14.** A surface, therefore, is not a part of a body. Lay, in thought, any number of surfaces one upon the other: in this way you would never obtain a body, but always again a surface. The common boundary which separates water from oil resting on it (*Fig. 5*), is neither water nor oil nor any other substance: it is not a body, but a surface. A surface is no more a part of a body than a shadow is a part of the wall on which it rests.

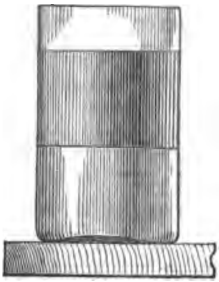

Fig. 5.

A line is no part either of a surface or of a body. Lay any number of lines side by side, the result is neither a surface nor a body, but always a line. The line which forms the common boundary between a red surface and a blue surface is itself neither red nor blue, nor has it any other color.

Lastly, a point is no part of a line, a surface, or a body. In conversation we often hear the word "point" used in the sense of a surface of a certain extent. Thus, we speak of the "point of crossing" of two or more roads. Substitute, in thought, for these roads, lines absolutely without breadth or thickness; then their intersection would be a point, as this word is understood in Geometry.

§ **15.** Except in thought, surfaces, lines, and points cannot exist apart from bodies. In thought, however, we can detach them, so to speak, from bodies, and study their forms and properties apart from the bodies to which they belong.

To aid the mind in reasoning about space conceptions, we *represent* them to the eye as follows:—

A point is represented by a fine dot made with a pencil on paper (*Fig. 6*), or with a crayon on the blackboard: and is *named* by means of a letter (A, B, C, etc.).

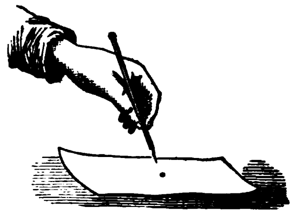

Fig. 6.

A line is represented by the fine continuous trace of a pencil on paper, or of a crayon on the blackboard, and named either by two letters, placed one at each of its ends (see *Fig. 7*), or by a single small letter (a, b, c, etc.).

A surface is represented and named by means of the lines which form its boundaries.

A body is represented and named by means of the surfaces which form its boundaries.

Exercises.—1. Give examples of bodies.

2. Is the space occupied by the room a body? Why?

3. Name the largest body you can; also the smallest.

4. Point out and name the dimensions of the room; this book; this pencil; this piece of rubber.

5. Name the dimensions of a board; a tower; a ditch; a well.

6. How many partial surfaces has the room? Point out and name the dimensions of each.

7. How many edges has the room? How many corners? Of what are the edges the intersections? the corners?

8. How many surfaces has this piece of rubber? Name their dimensions. Which are equal in magnitude?

9. A glass tumbler has an exterior (outer) and an interior (inner) surface. Give other examples of such bodies.

10. How many edges and corners has a pane of glass?

11. Cut twice through this body (say, an apple) so as to divide it (if possible) into two parts; three parts; four parts; five parts.

12. Are most natural objects (for example, stones, trees, leaves, flowers, animals) simple (like the cube, prism, etc.) or complex in shape?

13. When we use such an expression as "a shapeless mass," do we mean that the mass has no shape at all? What do we really mean?

14. In the process of beating gold, leaves are obtained not over $\frac{1}{300000}$ of an inch thick. Are such leaves surfaces or bodies?

15. In conversation, we often hear a string, a telegraph wire, etc., spoken of as *lines*. What are they, strictly speaking? Why?

16. Is the trace of a pencil-point on paper, which we call a *line*, really a line? What is it? Why?

17. When we make a point on paper with the end of a pencil, *is* it really a point, or does it *stand* for or *represent* a point? What is it, strictly speaking? Why?

V. — Generation of Space Magnitudes.

§ **16.** When we move the point of a pencil on paper, it leaves behind a trace which we call a line: reduce, in thought, the point of the pencil to a point as understood in Geometry, then its trace or *path* would be a geometrical line. Again, every one has observed that the path of the red-hot end of an iron rod, when moved rapidly in the dark, presents the appearance of a luminous line.

The path of a moving point is always a line: in other words, when a point moves in space it *describes* or *generates* a line.

When a line moves in space, it usually generates a surface. This fact may be illustrated by moving a crayon over the surface of the blackboard, with its longest side pressed against the board.

When a surface moves in space, it usually generates a solid. We may illustrate this truth by taking a thin piece of board (or a slate) to represent a surface, and pressing it down through a soft substance, like snow: the *hole* thus made is the path of the surface, and is a geometrical body or solid.

We have then, in all, three general truths or laws: —

I. — *The path of a moving point is a line.*

II. — *The path of a moving line is a surface.*

III. — *The path of a moving surface is a solid.*

NOTE. — Practically speaking, in Law I. *material points* are always meant; that is, *bodies which seem to be points, or are for the time regarded as points.* The stars, to our eyes, are material points; in reality, they are bodies of enormous size.

Exercises. — 1. What exception is there to Law II.? Illustrate with a pencil.

2. What exception is there to Law III.? Illustrate with a slate.

3. What is the path of a moving solid?

VI. — Straight and Curved Lines.

§ **17.** If a point move always in the same direction, its path is a *straight* line; if, on the contrary, the direction of its motion is continually changing, its path is a *curved* line or *curve*. In *Fig.* 7, *A B* is a straight line, *C D* is a curve.

Definitions.—I. A STRAIGHT LINE *is a line which has everywhere the same direction.*

II. A CURVED LINE, *or* CURVE, *is a line whose direction is continually changing.*

A line like *E F* (*Fig.* 7), composed of several straight lines, is called a *broken* line: and a line like *G H*, composed of straight and curved lines, is called a *composite* line.

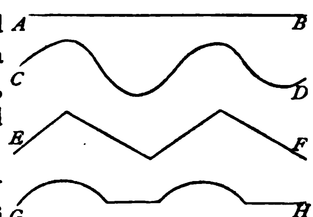

Fig. 7.

The figures on carpets and wall-papers, and the ornamental designs employed on articles of furniture and in architecture, furnish examples of broken and composite lines in endless variety.

The word *line*, when used alone, usually means a *straight line.*

Exercises. — 1. On the blackboard are lines *a, b, c, d,* etc. What kind of a line is *a?* Why? *b? c? d?* etc. Give the reason in each case.

2. Draw and name with letters two lines of each kind. In what two ways may you name or denote a line?

3. What kind of lines are the edges of the cube? the prism? the pyramid? the cylinder? this table? this ruler? this rubber? the room? also the rim of a tumbler? the spokes of a carriage-wheel? its tire? a telegraph wire? the letters M, N, O, U, V, Z?

Fig. 8.

4. What sort of a line does a flash of lightning often resemble? (*Fig. 8.*)

5. What is the path of a falling apple? a ball thrown into the air? the end of a watch-hand? the end of a pendulum? a drop of rain? a flying bird? a point on the tire of a carriage-wheel?

6. Give other examples of straight lines and curves.

7. In these exercises have we been using the word "line" in its strict geometrical sense? Why?

VII. — Plane and Curved Surfaces.

§ **18.** The faces of the cube are *flat* or *plane:* the side of a cylinder is *round* or *curved.* How can we define these two classes of surfaces?

Definitions. — I. A Plane Surface, *or* Plane, *is a surface on which straight lines can be drawn in all directions.*

II. A Curved Surface *is a surface on which straight lines cannot be drawn in all directions.*

A straight line which moves in space, with one end fixed and the other end sliding along another straight line, generates a plane surface; but if the second line is a curve, the surface generated is in general a curved surface.

If, for example, the second line is the circumference of a circle, the surface generated will be that of the cone (*Fig. 4*), except when the moving line is in the plane of the circle: in this case the surface generated will be a plane surface.

Test of a Plane Surface. — Apply to the surface, in different directions, the straight edge of a ruler; if the surface is plane, the edge of the ruler will always touch the surface from end to end.

Exercises. — **1.** On a tin pail point out a plane surface; a curved surface.

2. What kind of surfaces are the faces of a pyramid? the side of a cone? its base? the surface of the sphere?

3. Are the walls of the room plane or curved? the surface of the table? of a lamp-shade? of a hat? of a mirror? of a wash-basin? of the water in the basin? of a flower? of the ocean?

4. Give other examples of plane and curved surfaces.

5. Can you draw a straight line on the side of a cylinder? on the side of a cone? on the surface of a sphere?

6. Test whether the surface of your desk is a perfect plane.

VIII. — *Cornered and Curved Bodies.*

§ **19.** The cube and the bodies shown in *Fig. 2* are examples of *cornered* bodies. The cylinder and the bodies shown in *Fig. 4* are instances of *curved* bodies.

Definitions. — I. A Cornered Body, *or* Polyhedron, *is a body bounded entirely by plane surfaces.*

II. A Curved Body *is a body bounded partly or wholly by curved surfaces.*

Exercises. — 1. Give examples (*a*) of polyhedrons, (*b*) of curved bodies.
2. To which class does the earth on which we live belong?

IX. — Geometry.

§ **20.** *The Science of space conceptions is called* GEOMETRY.

The aim of Geometry is threefold: —

I. — *To discover the various properties of space figures arising from their having different forms.*

II. — *To show how to draw or construct space figures.*

III. — *To show how space magnitudes may be measured.*

Geometry is usually divided into Plane Geometry and Solid Geometry.

PLANE GEOMETRY *treats of space conceptions confined to one and the same plane.*

SOLID GEOMETRY *treats of space conceptions not confined to one plane* (*for example, solids and curved surfaces*).

Chapters II.–X. of this work treat of Plane Geometry; the remaining chapters, of Solid Geometry.

NOTES. — 1. The word *Geometry* comes from two Greek words meaning *land-measuring*. In ancient times, when the science was in its infancy, it consisted entirely of a few practical rules employed in measuring land, and used especially by the Egyptians in settling disputes about boundary lines destroyed by the annual overflow of the river Nile. Hence the name Geometry (or land-measuring) was given to the science by the Greeks.

2. No science excels Geometry in the number and importance of its practical applications. This fact is strikingly seen in the power which it gives us of *measuring* space magnitudes. If one asks: How much more will grow on this field than on that one? how much must this ship be loaded to sink ten feet into water? how to find the right way across the pathless ocean? how maps of the land and sea are made? how the distance of the moon can be found? at precisely what instant an eclipse of the sun will take place? when a comet will again be seen? how a cannon must be pointed in order that the ball may hit a certain mark? — these and similar questions, only a man skilled in Geometry can answer.

REVIEW OF CHAPTER I.

QUESTIONS.

1. How many dimensions has the cube, and what are they called?
2. How many faces, edges, and corners has the cube?
3. How many faces meet in an edge? how many edges in a corner?
4. Are the faces equal or unequal? the edges?
5. Answer the preceding questions for each of the bodies in *Fig. 2*.
6. What account can you give of the surfaces and edges of the bodies in *Figs. 3* and *4?*
7. Name a body which has no corners; also one which has neither edges nor corners.
8. Define a *solid*, a *surface*, a *line*, a *point;* how many dimensions has each, and what are they called?
9. Define the *intersection* of surfaces or of lines. What is the intersection of a surface? of a line?
10. What are *space conceptions? space magnitudes? space figures?*
11. What are the parts of a body? the parts of a surface? the parts of a line?
12. Illustrate the facts, that a surface is no part of a body, and that a line is no part of a surface.
13. Illustrate what is meant by a point in Geometry.
14. How are space conceptions represented and named?
15. Illustrate how space magnitudes are generated by motion. What are the general laws? the exceptions to them?
16. Define the four kinds of lines, and give examples.
17. Define *plane* and *curved* surfaces. How is a plane surface tested?
18. Show how a line must move to generate a plane surface; a curved surface.
19. Define *polyhedrons* and *curved bodies*, and give examples.
20. What is *Geometry?* its chief *aims?* its chief *divisions?*

EXERCISES.

1. Give examples of bodies shaped like a prism, a pyramid, a cone, a sphere.
2. Name the dimensions of a telegraph wire, a monument, a river, a well.
3. Name the dimensions of a cylinder, (*a*) when standing on its base, (*b*) when resting on its side.
4. How can you cut a prism in two so that each part will also be a prism?

5. How can you place two cylinders having equal bases together so as to form another cylinder?
6. Can you place two cubes together so as to form a new cube?
7. Here are several equal cubes: put them together so as to form a new cube. How many cubes does it take? How much larger is the new cube than one of its parts? its face than the face of one of its parts? its edge than the edge of one of its parts?
8. Mention bodies of *regular* shape, but differing in shape, from those thus far examined.
9. What kind of space magnitude is the human skin?
10. Give an instance where more than two lines intersect in the same point.
11. Can more than two surfaces intersect in the same line? Can you give an illustration?
12. When will a large body *appear* very small? Give examples.
13. Draw (if possible) a line with no ends; with one end; with two ends; with more than two ends.
14. If you slide a straight line along the two rails of a railroad, what kind of a surface would be generated?
15. If you move a straight line around the tires of two carriage-wheels on the same axle (always keeping it in contact with both tires), what kind of a surface would be generated?

CHAPTER II.

STRAIGHT LINES.

I. — Direction of One Line.

§ **21.** A straight line, AB (*Fig. 9*), may be regarded as having either the direction from A towards B, or the *opposite* direction from B towards A; in the first case, we should call it the line AB, in the second case, the line BA. Sometimes (as, for instance, on a guide-post) an arrow-head or a hand is used to point out which direction is intended; but, in general, we may assign to the line either direction.

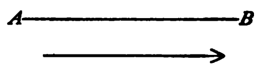

Fig. 9.

§ **22.** If we know only the direction of a straight line, we cannot determine its position or *locate* it; for, through different points, A, B, C, D, (*Fig. 10*), different straight lines may be drawn, all having the same direction.

Likewise, if we know only that a straight line passes through a known or *given* point, A, the line is not known; for, through a point any number of straight lines may be drawn.

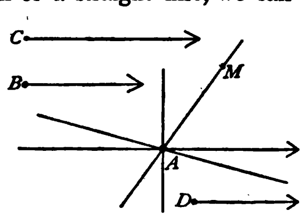

Fig. 10.

But through the point A, in a given direction, as that of the arrow, only *one* straight line can be drawn. And, also, through any two points, as A and M, but *one* straight line can be drawn.

These two truths may be stated as follows, in a general form, without reference to *Fig. 10*, or to any particular case :—

I.—*Through a given point in a given direction only one straight line can be drawn.*

II.—*Through two given points only one straight line can be drawn.*

Hence, if *one* point and the *direction* of a line are given, or if *two* points of the line are given, the line is *determined in position.*

§ **23.** For drawing straight lines on paper, or on the blackboard, a Ruler (*Fig. 11*)[1] with a straight edge is employed.

In order to draw a line on paper through two given points, lay the ruler on the paper so that its edge just touches the two points; then draw the line with a hard, well-sharpened pencil, holding the pencil nearly vertical, and always in contact with the edge of the ruler.

Fig. 11.

When we draw a line without the aid of any instrument to guide the hand, we are said to draw it FREE-HAND.

Exercises.—**1.** Draw four lines through a point, and name them with letters.

2. Make two points, and then join them by a straight line.

NOTE.—If A and B are two points, the expression *join A B* is to be understood as meaning *draw a straight line from A to B.*

3. Make three points not in the same line; then join them by straight lines. How many straight lines in all can be drawn?

4. How many straight lines in this way can be drawn between four points? five points? six points?

5. Draw, *free-hand*, a straight line; then correct or *rectify* it with the aid of a ruler. Repeat the exercise several times.

[1] The divided rule (see § **42**), which the pupil requires for other purposes, will also serve for drawing straight lines.

6. How can you test whether the edge of a ruler is straight?

Ans.—Draw a line with the ruler through any two points; then turn the ruler end for end, and, along the same edge, draw a line again through the same two points. If the edge is straight, the two lines will coincide and form a single line. (See § **22**, II.).

7. Test the edge of your ruler or divided rule.

8. Test the edge of the ruler at the blackboard.

9. A farmer has a four-sided field, and wishes to run a straight fence between two opposite corners. Explain how, with a person to assist him, he can find points on the proposed line of the fence between the corners.

Suggestion.—If the farmer stands a little way behind one corner, and places his eye so that the *line of sight* passes through both corners, this line will coincide with the proposed line of the fence.

§ **24.** Among the directions which a straight line may have, there are two of more practical importance than the others.

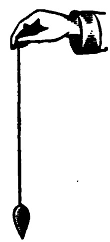
Fig. 12.

Fasten a weight to one end of a cord, and then hold the other end at rest in the hand: this forms what is called a *plumb* line. When the cord is at rest its direction is said to be VERTICAL. On the other hand, a pencil, stick, or similar object, when floating on the surface of still water, is said to have a HORIZONTAL direction.

Definitions.—I. A VERTICAL LINE *is a line which has the direction of a plumb line.*

II. A HORIZONTAL LINE *is a line which has the direction of any line in the surface of still water.*

III. *Lines neither vertical nor horizontal are called* INCLINED LINES.

Plane surfaces, also, are either vertical, horizontal, or inclined. Every plane in which a vertical line can be drawn, is a *vertical* plane; the surface of still water, or any plane similarly placed with reference to the earth's surface, is a *horizontal* plane; planes neither vertical nor horizontal are *inclined* planes.

The sides of the room, for example, are vertical planes; the

floor and ceiling are horizontal planes; the roof of the house is an inclined plane.

NOTES.—1. The surface of water, if of considerable extent (the ocean, for instance), is sensibly a curved surface, because the earth is round. This curvature makes itself evident on the seashore when we watch a ship sailing out of sight on the horizon: first the hull sinks out of sight; then the sails and lower parts of the masts; last of all, the tops of the masts. But on a pond three or four miles in length, the curvature is so small that it cannot be observed; hence, for all common purposes, the surface of the pond is regarded as a plane.

2. On paper draw vertical lines *towards* or *from* you; horizontal lines from *right* to *left*, or *left* to *right*.

Exercises.—1. Of the lines on the blackboard, which are vertical? which horizontal? and which inclined?

2. Draw a line on paper; then hold the paper so that the line shall be vertical. Is now the plane of the paper also vertical?

3. Hold a pencil vertical; horizontal; inclined.

4. Hold a book vertical; horizontal; inclined.

5. Draw three lines of each kind.

6. Draw a vertical line, mark five points on it, and through these points draw horizontal lines.

7. On the cube, point out vertical edges and faces; also horizontal edges and faces.

8. What directions have the edges of the prism? those of the pyramid? those of the cylinder?

9. What direction has a ladder when leaning against a wall? when lying on the ground? when suspended from one end by a rope? What direction have the *rounds* of the ladder in these three cases?

10. What direction has the mast of a ship? the path of a falling apple? the beam of a balance at rest? the surface of a lake? the side of a house? its roof?

11. At what time of the year are the sun's rays most nearly vertical? When are the sun's rays horizontal?

12. Give other examples of the vertical and horizontal lines and planes.

13. When is the minute-hand of a clock vertical? when horizontal? Answer the same questions for the hour-hand.

14. When two vertical planes (for example, the two sides of a room) intersect each other, what kind of a line is their intersection?

15. When a vertical plane intersects a horizontal plane what kind of a line is their intersection? Give an example.

16. Through a point how many vertical lines can be drawn? how many horizontal lines? how many inclined lines? how many planes of each kind?

17. In a vertical plane how many vertical lines can be drawn? how many horizontal lines?

18. In a horizontal plane how many vertical lines can be drawn? how many horizontal lines?

§ **25. Applications.**—I. In order to test whether a line or a plane (as a wooden beam, the wall of a house, etc.) is *vertical*, a PLUMB-RULE (*Fig. 13*) is often used. It consists of a piece of wood having two straight edges everywhere equally distant from each other. Near one end, exactly half way between the edges, a plumb-line is attached. Near the other end, also exactly half way between the edges, a mark is made. If we place one edge of the plumb-rule against a surface which is vertical, the line, when at rest, will cover the mark; if the surface is not vertical, the line will pass on one side of the mark. It is well to apply the test twice, using first one then the other edge of the plumb-rule; in both cases, the line ought to cover the mark.

Fig. 13.

II. To test whether a line or a plane surface is *horizontal* (or, as workmen say, *level*), instruments called LEVELS are used. The simplest kind of level is the *Plumb-line Level* (*Fig. 14*). It consists of three pieces of wood put together in the shape of the letter A, with a plumb-line suspended from the top. In order to adjust the instrument for use, a mark must be made on the cross-piece, where the plumb-line passes, when the feet of the level are

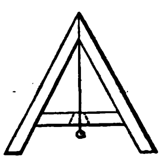

Fig. 14.

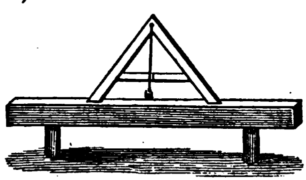

Fig. 15.

at the same height, or horizontal. This is done as follows: Place the feet of the level on any two points, and mark on the cross-bar

the place of the plumb-line; then turn the instrument end for end, rest it on the same points, and again mark the place of the plumb-line. The point midway between the two marks is the right one.

Fig. 15 shows the instrument in use to test whether a wooden beam is horizontal. As we see in the figure, the plumb-line passes a little to the left of the mark on the cross-bar. Which end of the beam is the highest?

In testing whether a *surface* (as that of a table) is horizontal, we must apply the level in at least *two* different directions, because a line in one direction on the surface might be horizontal while lines in other directions were inclined. But if two lines on the surface in different directions are horizontal, then the surface must be horizontal.

NOTES.—1. A common *Spirit-level* is shown in *Fig. 16*. It is more accurate than the plumb-line level, but (when properly made) more expensive. The essential part of the spirit-level is a glass tube, slightly curved, and filled with alcohol, except the space occupied by a bubble of air, which always seeks the highest part of the tube.

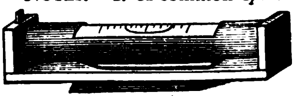

Fig. 16.

2. Surveyors and engineers, in establishing horizontal lines on the ground, or prolonging them, use levels mounted either on a staff which is driven into the ground, or on a three-legged stand (tripod) which rests on the ground. One may make a simple (but not very accurate) *plumb-line* level for use on the ground, as follows: Fasten a common carpenter's square (*Fig. 17*) in a slit in the top of a staff by means of a screw, and then tie a plumb-line at the angle so that it may hang beside one arm. When it has been made to do so by turning the square, the other arm will be horizontal.

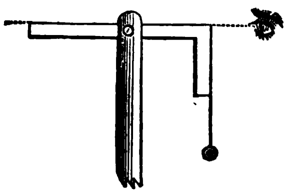

Fig. 17.

Exercises.—1. Choose some object in the room, and test whether it is vertical.

2. Choose some object in the room, and test whether it is horizontal.

3. Draw on the blackboard (with the aid of a plumb-rule) a vertical line.

4. Draw on the blackboard (with the aid of the plumb-line level) a horizontal line.

II.—Change of Direction. The Circle.

§ **26.** The direction of a straight line is *susceptible of change.* The hands of a clock, for instance, may be regarded as two lines which are constantly changing in direction. The minute-hand, in the course of an hour, points in all possible directions contained in the plane of the face of the clock.

When a straight line, AB (*Fig. 18*), moves without leaving the point A, until it has another direction, as AC, it is said to turn or ROTATE about the point A. If the motion continues, the rotating line will at length return to its first position, AB; it is then said to have made one REVOLUTION.

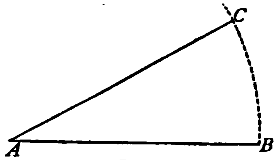

Fig. 18.

§ **27.** If, during the motion about A, the length of the line remains unchanged, its other end, B, will describe a *curved* line which returns into itself, and has the property that all its points are at the same distance from A. This curved line (*Fig. 19*) is called a CIRCUMFERENCE; the point A is called its CENTRE. Of all curved lines, the circumference is the simplest and the most important.

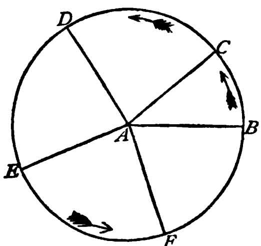

Fig. 19.

Definitions.—I. *A curved line which is everywhere equally distant from a point is called a* CIRCUMFERENCE.

II. *The point is called the* CENTRE.

III. *A part of a circumference is called an* ARC (*for example, BC or CD, Fig. 19*).

IV. *A straight line joining the centre to the circumference is called a* RADIUS [1] (*for example, A B or A C, Fig. 19*).

V. *The plane surface bounded by a circumference is called a* CIRCLE.

NOTE.—The word *circle* is also used in certain expressions in the sense of *circumference;* thus, in the expressions, "*to draw a circle,*" "*arcs of circles,*" strictly speaking, not the circle but the circumference or *boundary* of the circle is meant.

From definitions I. and IV. it follows that—

All radii of the same circle are equal in length.

Exercises.—**1.** Name objects on which circles and circumferences present themselves (ends of a cylinder, rim of a tea-cup, a hoop, etc.).

2. What does the tire of a carriage-wheel represent? the spokes? the axle? the part of the tire between two spokes?

3. In *Fig. 18*, as *A B* rotates about *A*, what kind of paths do points between *A* and *B* describe?

§ **28.** Circles (or, speaking more exactly, circumferences and arcs) are drawn on paper, etc., by the aid of an instrument called the COMPASSES or DIVIDERS [2] (*Fig. 20*).

The compasses (for use on paper) consist of two metal legs connected together by a pivot about which they can turn. The legs have fine, hard steel points. The lower part of one leg can be removed (by turning a screw on the side) and its place supplied by a *pencil*-leg, or a leg provided with a steel pen. In *Fig. 20* the pencil-leg is shown.

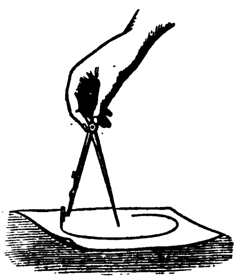

Fig. 20.

To draw a circle with a given radius, say two inches, fit the pencil-leg to the compasses, open the legs till the distance apart of their points, or *opening*, is just two inches, and then proceed as shown in *Fig. 20*.

[1] Plural, *radii*.

[2] So called from its use in dividing lines and angles into equal parts.

On the ground, circles are often described by the aid of a cord. When a gardener, for instance, wishes to trace a circumference, he proceeds as illustrated in *Fig. 21.*

Fig. 21.

Exercises.—1. Describe how to draw a circle of given radius on paper.

2. Describe how a gardener traces a circle on the ground.

3. Draw (on paper) a circle with a radius equal to the breadth of your ruler or divided rule.

4. Draw a circle with the large compasses on the blackboard.

5. Can you draw a circle on the blackboard with a radius greater than the greatest opening of the compasses?

6. Draw several arcs of circles.

7. Draw, free-hand, several circles. Which is drawn the best?

8. Draw three circles all having the same centre, or *concentric* circles, as they are called. The waves caused by throwing a stone into water are examples of concentric circles; what is their common centre?

9. Draw (*a*) any circle; (*b*) a circle with a point *A* as centre, and any radius; (*c*) a circle with a radius of one foot, and in any position (that is, having any point as centre); (*d*) a circle with the point *A* as centre, and a radius of one foot.

10. What two things completely determine the position and magnitude of a circle?

III.—Directions of Two Lines.

§ **29.** Two straight lines in the same plane must have either (*a*) the *same* or (*b*) *different* directions.

Case I. —When two different lines, as AB and CD (*Fig.* *22*), have the same direction, they are called PARALLEL lines. Bearing in mind what was said in § **21**, we may also regard one of them as having one direction, and the other the *opposite* direction. It is also evident that several lines, as AB, CD, EF, GH, may be parallel to one another.

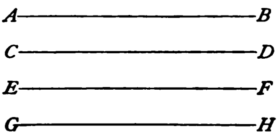

Fig. 22.

Definition.—PARALLEL LINES *are lines which have the same direction.*

That a line AB is parallel to a line CD is sometimes expressed, for the sake of brevity, by a sign; thus: $AB \parallel CD$.

The two rails of a railroad furnish a familiar illustration of parallel lines; substitute for the rails geometrical lines, and these would be parallel lines as understood in Geometry. The opposite edges of a table are another example of parallel lines.

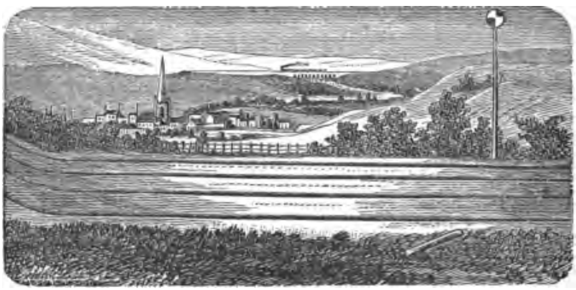

Fig. 23.

Every one has observed that the rails of a railroad course along without either approaching or separating. We may suppose them

to be prolonged as far as we please, without meeting. The same is true of the edges of the table and of all lines parallel to each other. In fact, if two parallel lines could meet when prolonged, they would then be two straight lines passing through the same point (the point of meeting), yet not coinciding; but this is impossible, for all straight lines drawn through the same point in the same direction must coincide and form only one line (see § **22**, I.). Therefore, —

Parallel lines can never meet, however far prolonged.

But two lines which would not meet if prolonged are not necessarily parallel. For instance: an edge of the table and one of the legs on the opposite side would never meet if prolonged, yet they are not parallel. The lines, in order to be parallel, ***must lie in the same plane.***

Two curves, also, may be parallel to each other. *Fig. 24* gives an example. The curves would not meet if prolonged, and, at *corresponding* points, as *A* and *B*, or *C* and *D*, they have the same direction. When a railroad makes a curve, the rails do not cease to be parallel. Concentric circles supply another example of parallel curves. (See *Fig. 190*, p. 212.)

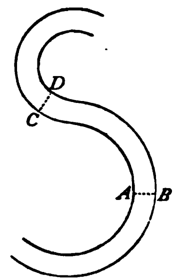

Fig. 24.

Two planes, or a plane and a straight line, are parallel, if they would not meet however far they were extended. The floor and ceiling of the room is an instance of parallel planes; the floor and an *edge* of the ceiling is an example of a plane and straight line parallel to each other.

Case II.—Far more common is the case where the two lines have different directions. Two such lines, as *A B* and *C D* (*Fig. 25*), either actually meet, or, if prolonged in one or the other direction, will approach

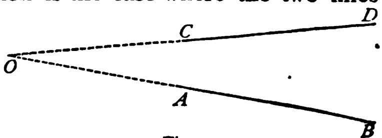

Fig. 25.

and at last meet in a point, O, called their *intersection* (§ **12**). Two edges of a table which meet at a corner are an example of lines having different directions; the corner is their intersection.

Exercises. — **1.** Which lines on the blackboard are parallel? which not parallel? Prolong the latter until they meet.

2. Can two lines, not parallel, intersect in more than one point?

3. How do you read the expression $AB \parallel CD$?

4. Hold two pencils (*a*) parallel; (*b*) so that they would intersect; (*c*) so that they are neither parallel nor would they intersect.

5. Hold two books (*a*) parallel; (*b*) not parallel.

6. Hold a pencil and a book (*a*) parallel; (*b*) not parallel.

7. Point out on the cube (*a*) parallel edges; (*b*) intersecting edges: (*c*) edges neither parallel nor intersecting.

8. Examine, in the same way, the edges of the prism and the pyramid.

9. Which edges of the room are parallel to the floor?

10. Give another instance of parallel lines; of parallel planes; of a line parallel to a plane.

11. Are two vertical lines parallel to each other?

Ans. — Strictly speaking, No; because, owing to the round shape of the earth, two vertical lines intersect at a point near the centre of the earth. But the distance to the earth's centre (nearly four thousand miles) is so great compared with the distance apart of two vertical lines, such as the edges of a house, etc., that the latter are always regarded as parallel.

12. Are horizontal lines always parallel to each other?

13. Can two vertical planes intersect each other?

14. Can two horizontal planes intersect each other?

15. Give an instance of parallel lines which are vertical; horizontal; inclined.

§ **30.** The simplest way to draw lines parallel to each other (especially when several are desired) is illustrated in *Fig. 26*. Besides a ruler, a piece of wood with three smooth, straight edges called a SQUARE, is needed. Keep the ruler at rest, slide the square along the edge of the ruler, and, in different positions of the square, draw lines along its edge; these lines will be parallel to each other. To draw a line through the point M, parallel to the

line PQ, place the ruler and square as shown in the figure, then slide the square along till its edge just touches M; the line drawn through M, along the edge of the square, will be the parallel required.

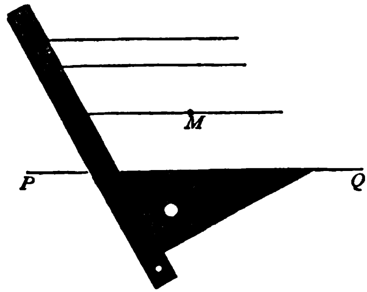

Fig. 26.

NOTE.—How parallel lines are drawn with ruler and compasses will be explained later. (See § **78**).

Exercises.—1. Draw a straight line; and then, through a point not on the line, a parallel to it.

2. Draw a straight line, and, through a point on the line, a parallel.

3. Draw five parallel lines about equally distant from each other.

4. Draw free-hand several parallel lines. Test and correct them with ruler and square.

IV.—Directions of Three Lines.

§ **31.** Three straight lines in the same plane present, as regards direction, *four* cases,—

(*a*) The three lines are all parallel;

(*b*) Two of the lines are parallel;

(*c*) No two lines are parallel, and all intersect in one point;

(*d*) No two lines are parallel, and they do not all intersect in a point.

These cases are all illustrated in *Fig. 27*. When necessary, the

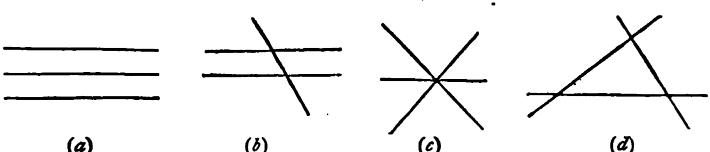

Fig. 27.

lines must be prolonged until they intersect. In case (*d*) the lines (prolonged, if necessary) intersect in three points, and com-

pletely enclose a portion of a plane surface, which is called a TRIANGLE.

NOTE.—The properties of the triangle will be studied in Chapter IV.

Exercises.—1. Draw any three lines, and then prolong them (by dotted lines) till they intersect.

2. In how many points can four lines intersect one another? (*a*) when all four lines are parallel? (*b*) when three are parallel? (*c*) when two are parallel? (*d*) when no two are parallel?

3. Examine five lines in the same way.

V.—Length of a Line.

§ **32.** The straight line which passes through two points, A and B (*Fig. 28*), is shorter than any other line passing through the points, and hence its length is taken as the DISTANCE of the points from each other.

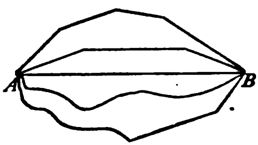

Fig. 28.

We all learn as soon as we begin to walk and to move about from place to place, that *a straight line is the shortest distance between two points.*

Illustration.—We may illustrate this truth as follows: Fasten one end of a string to a nail which is fixed in a wall, and pass the other end through a ring also attached to the wall. Now pull the

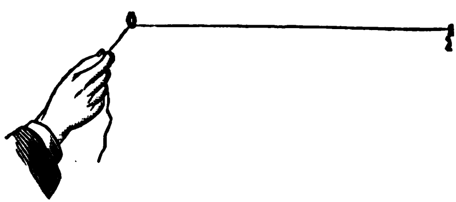

Fig. 29.

free end with one hand, and the part of the string between the wall and the nail will diminish in length until it is straight; and then it cannot be made any shorter.

Application. — Sign-painters, carpenters, etc., make use of this property when they wish to trace straight lines on wood. They chalk a cord and stretch it tightly between the points through which the line is to pass. Then seizing the cord by the middle, they

Fig. 30.

draw it away a little from the wood and then let it go. It springs back, strikes the wood a sharp blow, and leaves on it a white trace, which is a straight line.

§ **33.** As regards length, two lines must be either *equal* or *unequal.*

Definition. — *Two straight lines are* EQUAL, *if, when laid one upon the other, their ends or extremities can be made to coincide. If this cannot be done the lines are* UNEQUAL.

Expression of Equality. — If two lines, AB and CD, are equal, we express the fact thus: $AB = CD$. This expression is called an EQUATION, and is read: the line AB is equal to the line CD. The sign $=$ is called the SIGN OF EQUALITY.

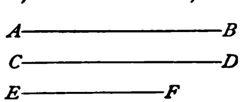

Fig. 31.

Expression of Inequality. — The lines AB and EF (*Fig. 31*) are unequal lines. *AB is greater than EF,* or *EF is less than AB.* In place of the words in italics, we often employ the signs $>$ and $<$, and write: $AB > EF$, and $EF < AB$.

Exercise. — Read the following: $a = b$; $c > a$; $x < y$; $AB < PQ$; $RS > PQ$; $EF = HK$; $CD > HK$; $XY < CD$.

§ **34.** In order to test the equality or inequality of two lines, it is not necessary to place one line upon the other, as the definition of the last section might seem to require. We find it usually more convenient to compare both lines with a *third* line, which can be readily placed upon or by the side of each of them.

In constructing geometrical figures and diagrams, we make this comparison (especially when great accuracy is required) by means of the dividers (compasses). Suppose, for example, that we wish to test whether the lines AB and CD (*Fig. 31*) are equal. Open the dividers, and place one point on A, the other on B; this is called *taking the distance AB between the points of the dividers.* Then, taking care not to alter the opening of the dividers, place one point on C and observe whether the other point will fall on D; if it does fall on D, we know that $CD = AB$.

NOTE. — In general, lines are compared in length by *measuring* them, and expressing their lengths in feet, miles, metres, etc., as will be explained in Parts VII. and VIII. of this chapter.

Exercises. — 1. Draw two lines, and then test whether they are equal.

2. Draw a horizontal line; then draw a vertical line equal to it.

3. Explain how to make a line equal to a given line.

4. Draw a line equal to one edge of this cube.

5. Draw lines equal to the different edges of this prism.

6. Draw four lines, a, b, c, d, as follows: $a = b$, $c > a$, $d < b$.

7. Draw any four lines, m, n, o, p; then test their lengths, and write the results with the proper signs (=, or >, or <).

8. Try to draw, free-hand, two *equal* lines, one horizontal, the other vertical; then test their equality.

9. Draw two lines, each longer than the greatest opening of the dividers. How can you test whether they are equal or not?

§ **35.** In testing, with the dividers, the lengths of two lines, the third line with which we compare each of them is the *distance* between the points of the dividers. We *assume* that if the two lines are each equal to this distance, they must be equal to each other. This assumption is so very obvious that it almost escapes notice.

If two lines — or any two magnitudes — are each equal to a third, of *course* they are equal to each other. In fact, this is a truth so simple in its nature that it cannot be made clearer by trying to prove it. Truths of this kind — that is to say, *self-evident* truths — are called AXIOMS.

Besides the axiom here under consideration, there are four others often useful in Geometry. We shall now state them all, and shall refer to them hereafter as here numbered.

Axioms. — I. ***If two magnitudes are each equal to a third, they are equal to each other.***

II. ***If equals are added to equals, the sums are equal.***

III. ***If equals are subtracted from equals, the remainders are equal.***

IV. ***If equals are multiplied by equals, the products are equal.***

V. ***If equals are divided by equals, the quotients are equal.***

§ **36.** The length of a line (like its direction) is susceptible of change. We may suppose it either to *increase* or to *diminish*.

Prolong the line AB (*Fig. 32*) to any point C; then the line AC is equal to the *sum* of the lines AB and BC; here we may employ to advantage the sign $+$ of addition, and write, —

Fig. 32.

$$AC = AB + BC.$$

Conversely, the line AB is the *difference* between the lines AC and BC; or we may use the sign $-$ of subtraction, and write, —

$$AB = AC - BC.$$

Prolong a line AB (*Fig. 33*) by adding to it repeatedly (by means of the dividers) lines BC, CD, DE, etc., each equal to AB. Then AC is twice as long as AB, AD is three times as long, AE four times as long, etc. That is, we have *multiplied* the

line AB by 2, 3, 4, etc. It is usual to omit the sign × of multiplication, and write the results thus: $AC = 2\,AB$, $AD = 3\,AB$, $AE = 4\,AB$, etc. Also $AE = 2\,AC$, $AL = 5\,AC = 2\,AF$.

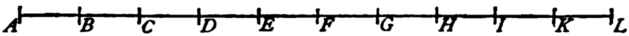

Fig. 33.

Conversely, a line can always be *divided* into equal parts. In the above line, for example, AB is one-half of AC, one third of AD, one-fourth of AE, etc.; in other words, —

$$AB = \frac{AC}{2} = \frac{AD}{3} = \frac{AE}{4}, \text{ etc. Also, } AC = \frac{AG}{3},\ AE = \frac{AI}{2}.$$

These results may be summed up in the following general truth: *Lines, like numbers, may be added, subtracted, multiplied, and divided.*

Remark. — Observe, however, that we have only multiplied and divided a *line* by a *number*, and that the result has always been *another line*. If, for instance, we multiply the line AB by the number 10, we simply repeat the line a certain number of times, and the result is the line AL; or, writing the operation in the form of an equation, —

$$10 \times AB = AL.$$

In the converse operation of division there are two cases: —

(*a*) We may divide the *line* AL by the *number* 10, obtaining as the quotient the *line* AB. In this case we *divide the line into a certain number of equal parts.*

(*b*) Or, we may divide the *line* AL by the *line* AB, and now our quotient will be the *number* 10. In this case we *find how many times one line is contained in the other line.* In this sense division is the *comparison, as regards magnitude, of two things of the same kind,* and the quotient is their *relative magnitude* or *ratio.*

Exercises.—1. In *Fig. 33*, what line is equal to $BD+DE$? to $AE-AD$? to three times the sum of AB and BC? to four times the difference of AC and CD?

NOTE.—Perform the following exercises with ruler and dividers, and *not* with the aid of the divisions of a divided rule.

2. Draw two lines; then draw lines equal (*a*) to their sum, (*b*) to their difference.

3. Draw $AB=CD$, and $MN=PQ$; then draw lines equal to $AB+MN$, $CD+PQ$, $AB-MN$, $CD-PQ$. Why are the first two of these lines equal? Why are the last two also equal?

4. Draw a line three times as long as your pencil.

5. Draw a line, AB, and prolong it to a point, P, so that $AP=5AB$.

6. Explain how to repeat or multiply a line any number of times.

7. Draw a line equal in length to the sum of the edges which bound one face of this cube.

8. Draw a line; then divide it into two, four, and eight equal parts.

NOTE.—This is to be done here by *repeated trial* with the dividers.

9. Divide a line, free-hand, into five equal parts. Test and correct with the dividers. Repeat this exercise several times.

VI.—Ratio of Two Lines.

§ 37. Definitions.—I. *If a line is contained in another line one or more times without a remainder, the first line is called a* MEASURE, *or* ALIQUOT PART, *of the second; and the second line is called a* MULTIPLE *of the first.*

II. A COMMON MEASURE *of two or more lines is a line which is exactly contained in each of them.*

III. *The* RATIO *of two numbers is the quotient obtained by dividing one number by the other; in other words, it expresses how often one of the numbers will contain the other.*

IV. *The* RATIO *of two lines is the number of times one of the lines will contain the other; it is equal to the ratio of the numbers which express how often a common measure is contained in each of the lines.*

V. *Of the two terms of a ratio, the first is called the* ANTECEDENT, *the second is called the* CONSEQUENT.

For example: in *Fig. 34*, where $AB = BC = CD = DE = EF$, the line AB is a *measure* of AB, of AC, of CF, in short, of every line represented in the figure; therefore AB is a *common measure* of all these lines. On the other hand, all the lines in the figure are *multiples* of AB.

Fig. 34.

If we compare the two lines AB and AF, we see that their common measure AB is contained once in AB and five times in AF; hence, the lines AB and AF are to each other as the numbers 1 and 5; that is, they have the *ratio* 1 : 5. By expressing this ratio as a fraction, we obtain the equation, —

$$\frac{AB}{AF} = \frac{1 \times AB}{5 \times AB} = \frac{1}{5}.$$

The *reciprocal* ratio of AF to AB is that of 5 to 1, or simply 5.

NOTE. — In common language, instead of using the term ratio, we say, AB is one-fifth as long as AF; or, reciprocally, AF is five times as long as AB.

In like manner, it is easy to see from *Fig. 34* that —

The lines		AB and AC	have the ratio		1 : 2;
"	"	AC and AB	"	"	2 : 1;
"	"	BD and AF	"	"	2 : 5;
"	"	CF and AE	"	"	3 : 4; etc.

In the form of fractions, these ratios would be written, —

$$\frac{AB}{AC} = \frac{1}{2};\ \frac{AC}{AB} = \frac{2}{1} = 2;\ \frac{BD}{AF} = \frac{2}{5};\ \frac{CF}{AE} = \frac{3}{4};\ \text{etc.}$$

Exercise. — What is the ratio of AD and AB (*Fig. 34*)? AD and AC? AD and AD? AD and AE? AD and AF? AC and AE? AE and AF? Write the values both with dots (:) and as fractions. In these ratios, which term is the antecedent? which the consequent?

§ **38.** In order to find the ratio of two lines, apply the shorter line to the longer line as often as possible; if the shorter line is a

measure of the longer,—if, for example, it is contained in the longer exactly five times,—then 1 : 5 is the ratio of the shorter line to the longer.

If, however, the shorter line is not a measure of the longer, the problem becomes more difficult. After we have applied the shorter line to the other line as many times as possible, there is a remainder less than the shorter line. For example: in the case of the lines MN and PQ (*Fig. 35*), when MN has been applied twice from P towards Q, there remains the portion RQ of the longer line. To meet this difficulty, apply RQ to MN as often as possible; then apply the new remainder SN to RQ, which contains it exactly twice.

This series of operations is precisely like the process of finding the greatest common divisor of two numbers; it is, in fact, the method of finding the *greatest common measure* of two lines.

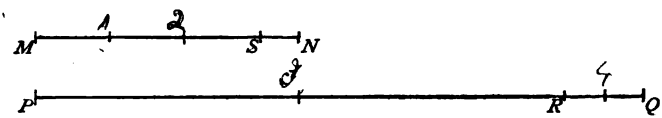

Fig. 35.

From what precedes we see that—

$$RQ = 2\,SN;$$

$$MN = 3\,RQ + SN = 6\,SN + SN = 7\,SN;$$

$$PQ = 2\,MN + RQ = 14\,SN + 2\,SN = 16\,SN;$$

hence, $\dfrac{PQ}{MN} = \dfrac{16\,SN}{7\,SN} = \dfrac{16}{7}$; and $\dfrac{MN}{PQ} = \dfrac{7\,SN}{16\,SN} = \dfrac{7}{16}$.

SN is the greatest common measure of the two lines; the longer contains it sixteen times, the other seven times; the ratio of the lines, therefore, is that of the numbers 16 and 7, either 16 : 7 or 7 : 16.

It may happen that we cannot find a common measure of two lines by this process, for the reason that the lines have no common measure; such lines are called INCOMMENSURABLE LINES.

Remark. — The exact ratio of two incommensurable lines cannot be expressed by numbers; but by dividing one of the lines into a great number of equal parts, and applying one of these parts to the other line as often as possible, and neglecting the remainder (which will be very small compared with either of the lines) we are able to *approximate* very nearly to the true value of the ratio.

If, for instance, we divide one of two lines into 1000 equal parts, and find that the other line contains more than 600, but less than 601, of these parts, we know that the true value of the ratio lies between $\frac{600}{1000}$ and $\frac{601}{1000}$. For nearly all purposes we should assume that the ratio was $\frac{600}{1000}$, or $\frac{3}{5}$; in other words, we should assume that one line was three-fifths as long as the other.

Exercises. — **1.** In *Fig. 36*, of what lines is MN a common measure? How many times is it contained in each of them? Of what lines is PQ $(=\frac{1}{4}MN)$ a common measure? How many times is it contained in each?

2. Express the values of the other lines in terms of PQ.

Ans.—$MN=4\,PQ$; $AB=24\,PQ$; $CD=16\,PQ$; $EF=17\,PQ$.

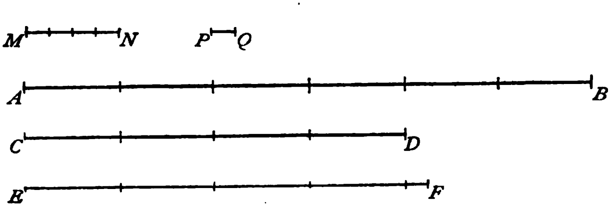

Fig. 36.

3. How many times is $\frac{1}{2}\,PQ$ contained in each line in *Fig. 36?* Is it a common measure of all of them? If we divide PQ into three, four, five, etc., equal parts, will one of the parts be a common measure of all the lines?

4. Of what lines is $2\,PQ$ a common measure? $2\,MN$?

5. What is the greatest common measure of MN and AB? of AB and CD? of MN, AB and CD? of MN, AB, CD and EF?

6. Find the ratio of PQ to each of the other lines in *Fig. 36*, and write the results both fractionally and with dots.

7. Find the ratio of MN to all the other lines, and write the results both fractionally and with dots.

8. Find the ratio of AB to the other lines, and write out the results.

9. What is the ratio of CD to EF? What is the reciprocal ratio?

10. Draw any two lines, and then find their ratio.

VII.—*Units of Length.*

§ **39.** In comparing the lengths of lines with one another a great amount of time and trouble is saved by employing *units* of length.

Definitions.—I. A UNIT *of length is a known length which men agree to employ as a common term of comparison, or common measure for all lines.*

Units of length receive individual names, as the *foot*, the *yard*, the *meter*, the *mile*, etc.

II. *To* MEASURE *a line is to find how often it will contain a unit of length.*

In other words, to measure a line is to find its ratio to the unit of length.

III. *The number of times a line contains a unit of length is the* NUMERICAL VALUE *of the line with reference to that unit; joined to the name of the unit, the whole expression is called the* LENGTH *of the line.*

Examples.—6 feet, 9 meters, 12 miles.

NOTE.—In olden times, units of length were usually taken from some part of the human body. The word *foot* indicates its origin. The *cubit* was the length of the forearm, from the elbow to the end of the middle finger. The first attempt to establish a standard *yard* in England was in 1120, in the reign of King Henry I., who ordered that the yard, the ancient ell (Latin *ulna*, elbow, arm), should be of the exact length of his own arm. At the present day, horses are described as so many "*hands* high," and a place is sometimes said to be so many "*paces* distant." The *span*, the *palm*, and the *fathom* all derive their origin from the human body; the fathom was the distance between the extreme points of the outstretched arms. But the units of length now recognized by law among civilized nations are defined by the State with the utmost precision, by reference to a standard made of platinum or other hard metal, which suffers no sensible change from age to age. This standard is kept at the capital of the nation, and carefully made copies of it are preserved in public buildings in the chief cities where the unit is employed.

§ **40.** The units of length established by law, and used for most purposes in the United States and in Great Britain, are the INCH, the FOOT, the YARD, the ROD, the MILE.

12 inches (in.) = 1 foot (ft.).	$5\frac{1}{2}$ yards = 1 rod.
3 feet = 1 yard (yd.).	320 rods = 1 mile.

NOTE.—The *geographical* or *sea* mile is one-sixtieth of a degree of a meridian of the earth and is equal to 1.15 *common* or *statute* miles, very nearly.

Exercise. — How many inches in one yard? in one rod? in one mile?

§ **41.** The units of length now employed by the people of most civilized nations, and by the men of science of all nations, are called METRIC units, from the name of the fundamental unit, the meter.[1] They form, taken together, a part of the Metric System of Weights and Measures, first adopted by the French just after the great revolution of 1792.

Definition. — *The* METER — *the fundamental unit from which all other units in the metric system are derived — is defined by law as the length of a certain platinum bar kept at Paris.*

NOTES.—1. The nations which have adopted the metric system take from the Paris meter the most perfect copies which can be made, and in the capital of each nation are preserved a set of metric measures and weights with which those used by the people must agree.

2. The founders of the metric system defined the meter as *one ten-millionth part of the distance from the Equator to the North Pole.* Extraordinary efforts were made to construct a platinum bar which should have this exact value. An arc of a meridian passing through France and Spain was measured with extreme care, and by combining the results with other known facts the distance from the Equator to the Pole was calculated. The platinum bar based on these results was supposed at the time to be exactly what it was defined to be. But not long afterwards errors in the measurements were discovered, and it is now known that the Paris meter (the platinum bar) is less than its intended value by an amount not exceeding $\frac{1}{5590}$ of the whole.

The other units of length are multiples and divisions of the meter as follows: —

Ten meters are called one DEKAMETER.

One hundred meters are called one HEKTOMETER.

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[1] The word *meter* comes from a Greek word meaning *measure*, and is pronounced meé-ter. The French write the word *mètre*.

One thousand meters are called one KILOMETER.

One tenth of a meter is called one DECIMETER.

One hundredth of a meter is called a CENTIMETER.

One thousandth of a meter is called a MILLIMETER.

NOTES.—1. Ten thousand meters is also called a MYRIAMETER.

2. The names of the multiples *deka*meter, *hecto*meter, *kilo*meter, *myria*meter are formed by prefixing to the word meter the Greek numerals deka (10), hecto (100), kilo (1000), and myria (10000); for the divisions of the metre, the Latin numerals, *deci*, *centi*, and *milli* are used.

Abbreviations.—Meter = m; dekameter = dkm; hectometer = hm; kilometer = km; decimeter = dm; centimeter = cm; millimeter = mm.

TABULAR VIEW.

km.	hm.	dkm.	m.	dm.	cm.	mm.
1 =	10 =	100 =	1000 =	10000 =	100000 =	1000000
	1 =	10 =	100 =	1000 =	10000 =	100000
		1 =	10 =	100 =	1000 =	10000
			1 =	10 =	100 =	1000
				1 =	10 =	100
					1 =	10

This table shows that a given number of units of one value are reduced to units of the next *lower* value by *multiplying* by 10, and to units of the next *higher* value by *dividing* by 10.

Examples.—1. Reduce 32 meters to decimeters, and also to dekameters.
Ans. 320^{dm}; 3.2^{dkm}.

2. Reduce 0.764 of a kilometer to meters.

Solution.—Since we wish to reduce kilometers to the *third* lower unit, we must multiply by 10 *three* times. As in decimal fractions, this is done by moving the decimal point *three* places to the *right*. *Ans.* 764^{m}.

3. Reduce 376 millimeters to kilometers.

Solution.—Here we have to reduce millimeters to the *sixth* higher unit; therefore, we must *divide* by 10 *six* times. That is, move the decimal point *six* places to the *left*. *Ans.* 0.000376^{km}.

Remarks.—1. The units *most used* are, the kilometer (for long distances), the meter, and the centimeter and millimeter (for lengths less than one meter).

2. In practice, *only one unit* is employed to express a length. We say 76^{cm} or 760^{mm} (*not* $7^{dm}\ 6^{cm}$); 332^{m} (*not* $3^{hm}\ 3^{dkm}\ 2^{m}$, and *not* $33^{dkm}\ 2^{m}$), etc.

3. In reading the *fraction* of a unit, the word signifying the number of decimal places may be omitted; thus, 4.5^{m} may be read: four meters five; 4.52^{m} may be read: four meters fifty-two, etc. This is analogous to the common way of reading money; $2.75, for instance, is read: two dollars seventy-five (or, yet more simply: two seventy-five). In the case of money, the omitted word is cents; with metric units it is tenths, hundredths, thousandths, etc., according to the number of decimal places.

4. In *adding*, *subtracting*, etc., metric units, they must first all be reduced to a common unit, as, for example, the meter. Thus, —

$$6.4^{m} + 27^{cm} = 6.4^{m} + 0.27^{m} = 6.67^{m.}$$

COMPARISON OF METRIC AND ENGLISH UNITS.

1 kilometer	=	0.62137	miles	(about $\frac{5}{8}$ of a mile).
1 meter	=	1.0936	yards	(about $39\frac{1}{3}$ inches).
1 centimeter	=	0.3937	inch	(about $\frac{2}{5}$ of an inch).
1 millimeter	=	0.0394	inch	(about $\frac{1}{25}$ of an inch).
1 mile	=	1609	meters	(about $1\frac{3}{5}$ kilometers).
1 yard	=	0.9144	meter	(about $\frac{10}{11}$ meter).
1 foot	=	0.3048	meter	(about 30 centimeters).
1 inch	=	2.54	centimeters	(about $2\frac{1}{2}$ centimeters).

NOTE. — It might be well for the pupil to learn by heart the approximate values (not the others) just given in parentheses; but the best way to become familiar with the values of the meter and its divisions is by comparing a meterstick with a yardstick, by comparing the divisions on a divided rule, and by solving exercises like the following.

Exercises. — 1. Give the metric units of lengths thus: 10 millimeters make 1 centimeter, 10 centimeters make 1 decimeter, etc.

2. Write a tabular view of the metric units of length.

3. Write with abreviations: 8 meters; 72 kilometers; 324.6 dekameters; 87.34 hectometers; 9.27 centimeters; 4.2 decimeters; 13 millimeters.

4. Read the following: 37.5^{m}; 947^{mm}; 846.7^{cm}; 18^{dm}; 9.04^{km}; 45.206^{hm}; 77.96^{dkm}.

5. Reduce 8844^{mm} to meters and to kilometers.

6. Reduce 3.27^{km} to meters and to millimeters.

7. Reduce 12.4^{hm} to all the lower units.

8. Reduce 180^{cm} to all the higher units.

9. What part of a kilometer is 1^{m}? 1^{cm}? 1^{mm}?

10. 25^{cm} are what part of a meter? also what part of a kilometer?

11. $24.32^{m} + 6^{dm} + 18^{dm} + 260^{cm} =$ how many meters?

12. $4^{dkm} + 4^{m} + 4^{dm} + 4^{cm} + 4^{mm} =$ how many meters?

13. $6^{km} + 6^{hm} + 6^{dkm} + 6^{m} + 6^{dm} =$ how many kilometers?

14. A shopkeeper sells ribbon as follows: to the first customer 8^{m}, to the second 70^{cm}, to the third 4.5^{m}, to the fourth 2^{dm}, to the fifth 80^{cm}; how many meters does he sell in all?

15. What is the difference in length of two nails, one of which is 4^{cm}, the other 28^{mm} long?

16. From 10^{m} take 10^{mm}.

17. From 100^{km} take 100^{mm}.

18. $87^{cm} \times 36 =$ how many meters?

19. $42.64^{m} \times 80 =$ how many kilometers?

20. If the post-office is 900^{m} from my house, and I go to it twice a day for letters, how many kilometers do I walk in a week?

21. I receive an order for 20 rulers, each to be 30^{cm} long; what length of wood is required?

22. How many rails each 7.5^{m} long are required to build a track 300^{km} long?

23. How long would it take an express train going at the average rate of 40^{km} an hour to travel from Boston to San Francisco, a distance of 5500^{km}?

24. If 1^{m} of silk costs \$3.00, what costs 1^{dm}? 1^{cm}? 1^{mm}? 1^{dkm}? 1^{hm}? 1^{km}?

25. Reduce the following values to English equivalents: velocity of an express train $= 60^{km}$ an hour, height of Mt. Blanc $= 4815^{m}$, usual height of a barometer $= 76^{cm}$.

26. Reduce the following values to metric equivalents: velocity of light $= 190{,}000$ miles a second; distance from Boston to New York $= 236$ miles; average height of a man $= 5$ feet 8 inches.

27. Which is greater, 3^{km} or 2 miles? 3^{dm} or 1 foot?

28. What unit would you use to express the distance from New York to Chicago? the height of a mountain? the length of a room? the dimensions of a sheet of paper? the thickness of a soap-bubble?

29. Why is the Metric system also called a *decimal* system?

30. Which do you consider preferable, metric units or the foot, yard, etc.?

VIII.—Measuring a Line.

§ **42.** In order to measure a line various instruments are employed. Among them we may mention the Divided Rule, the Yardstick (or Meterstick), the Tape-Rule or Roulette, and the Surveyor's Chain.

Lines on paper are generally measured by placing beside them a divided rule. A very useful kind of divided rule is a rule one foot long, divided on one side into inches and parts of an inch, on the other into decimeters, centimeters, and millimeters. On the blackboard a yardstick or meterstick is usually employed.

Longer lines, such as the length of a room or of a garden, can be more rapidly measured by the help of a tape-rule or roulette. These are made of cloth or flexible steel, wound around an axis, and enclosed in a small case suitable for the pocket. They are made of various lengths up to one hundred feet.

In measuring lines on the ground surveyors employ a chain sixty-six feet long, containing one hundred links, or a steel tape one hundred feet long. French surveyors use a chain one dekameter long, containing fifty links. If a chain or steel tape is not at hand, a rope divided by knots into yards or meters may be employed.

In the absence of any better means of measuring a line, *pacing* must be resorted to. The average value of a pace must be found by counting how many paces are made in walking a certain known distance, as ten meters. Pacing is not a very exact way of measuring a line, but it is a rapid way, and is always at our disposal.

In many cases a line is not measured *directly*, but its length is *computed* from lines of known length by methods which will be described hereafter. Indeed, it often happens that the direct measurement of a line is impossible; for example, the breadth of a river, the height of a mountain, the diameter of the earth, and, in

general, the distance of an inaccessible object (as the sun or the moon).

Exercises. — 1. Make a line 3^{dm} long, and divide it into centimeters.

2. Draw lines with lengths as follows: 16^{cm}; 0.6^{dm}; 175^{mm}.

3. Draw lines with lengths as follows: 76^{cm}; 14^{dm}; 1.36^{m}.

4. Draw, free-hand, a line 3^{dm} long. Measure it with the divided rule, and find what error you have made. Repeat the exercise three times.

5. Measure, in metric units, with the divided rule, the following lines, and write out the results: (*a*) Length, breadth, and thickness of a book; (*b*) length and breadth of your desk; (*c*) length of your lead-pencil; (*d*) length of your middle finger.

6. Estimate first, by the eye, then measure with the meterstick, the height of the door.

7. Measure, with a roulette, the length and breadth of the room; also, the length and breadth of the play-ground.

8. Find the value of your pace by pacing twenty meters.

9. Measure, by pacing, the distance from the schoolhouse to your home.

§ **43.** If we look on a map of a large country like the United States, we see that distances which amount to hundreds of miles are *reduced in length* and represented by lines at most a few inches long. It is clear that, in constructing maps, plans[1] of estates, etc., the actual lengths of lines must be very much reduced; for, otherwise, the map or plan would need to be as large as the country or the estate which it represents.

A divided line, or other means for enabling us to reduce lengths measured on the ground in a given constant ratio, is called a REDUCING SCALE; and the operation of drawing on paper lines whose length shall be one-half, one-fourth, one-tenth, or any other fractional part of the lines measured on the ground, is called DRAWING TO SCALE.

[1] Every one knows what a *map* is; a *plan* is a portion of a horizontal surface, or surface assumed as horizontal, represented on paper together with its subdivisions and other details of importance which it contains. A *diagram* is any set of lines drawn on paper, etc., to illustrate a statement or aid in a demonstration; the so-called *Figures* of Geometry are diagrams.

Suppose, for example, that we take as a reducing scale the divisions and subdivisions of a divided rule, and that lines are to be drawn to the scale of 1 meter to 500 meters; in other words, that the ratio or *scale* of reduction is 1 : 500. Then 1 meter would be represented on the paper by a line $\frac{1}{500}^{\text{m}} = 2^{\text{mm}}$ long, a distance of 48 meters by a line $\frac{48}{500}^{\text{m}} = 9.6^{\text{cm}}$ long; a length of 66.5 meters by a line $\frac{66.5}{500}^{\text{m}} = 13.3^{\text{cm}}$ long, etc. And, in general, the reduced length would be found by dividing the real length by 500.

Conversely, the real length of a line would be found from the reduced length by multiplying the reduced length by 500. Thus, a line 8 centimeters long on the paper, would represent a line which was in reality $8^{\text{cm}} \times 500 = 4000^{\text{cm}} = 40^{\text{m}}$ long.

Scales are named by expressing the ratio of reduction, either fractionally, thus, $\frac{1}{100}$, $\frac{1}{500}$, $\frac{1}{10000}$, etc., or with two dots, thus, 1 : 100, 1 : 500, 1 : 10000, etc.

A reducing scale is sometimes printed or drawn by the side of the map or diagram, and its divisions numbered in terms of the real lengths of the lines represented on the paper. We can then easily ascertain with the dividers the real length represented by a line on the paper. How?

The scale of reduction *must be the same* for all lines on the same map, plan, or diagram. If this rule were not followed, the appearance of two lines on the paper would give a false idea of their actual relative lengths; and, besides, the *shape* of the map or diagram would differ from the shape of what it professed to represent; in other words, the map or diagram would be *distorted*.

NOTES.—1. The choice of a scale depends on the real lengths of the lines compared with the size of the paper on which they are to be represented. The scale chosen should be such as will enable us to represent all the lines on the paper, and also leave a fair margin around the edge of the paper.

2. The following are examples of scales actually employed:—United-States Coast Survey, charts of small harbors, etc., 1 : 5000 down to 1 : 60000; charts of bays, sounds, etc., 1 : 80000 down to 1 : 200000; general charts, usually, 1 : 400000; Ordnance Survey of Great Britain, partly 1 inch to 1 mile (1 : 63360), and partly 6 inches to 1 mile (1 : 10560); Government Survey of France, original scale, 1 : 20000, scale of copies, 1 : 40000 and 1 : 80000.

3. For ordinary purposes, a reducing scale need not be made on the paper; we

choose the scale, then compute the reduced lengths, and then lay off these lengths on the paper with the aid of a divided rule. But when great accuracy is demanded, it is necessary to have on the paper a reducing scale by means of which we can determine the length of a line with extreme precision. Such a reducing scale is described in Chapter VIII. (See p. 184.)

Exercises.—**1.** Draw five parallel lines, and take on them to the scale 1 : 200 the distances 30^m, 20^m, 9^m, 48^m, 15^m.

2. Draw any three lines, and find what lengths they represent to the scale 1 : 200.

3. What is the value of the scale if 1^m is represented by 4^{cm}?

4. If 5^{cm} represent 100^m natural length, what length will represent 1^m? 20^m? 1^{dkm}? 12^{km}?

5. In what ratio are natural lengths reduced if the scale is ¼ inch to the foot? ½ inch to the mile? 2 millimeters to the kilometer?

Ans. 1 : 48; 1 : 126720; 1 : 500000.

6. The distance between two towns, drawn on paper to the scale 1 : 40000, is represented by a line 1^{dm} long; find the distance between the towns.

Ans. 4^{km}.

7. In a railroad survey the total length is eighty miles. What scale will enable us to represent a plan of the road on a piece of paper just one foot in length?

8. Draw on the blackboard three horizontal lines. Then represent them on paper, choosing a suitable scale for this purpose. Write the value of the scale on the paper under the lines.

REVIEW OF CHAPTER II.

QUESTIONS.

1. How many directions has a straight line?

2. How is a straight line determined in position?

3. How is a straight line drawn?

4. What is *free-hand* drawing?

5. What is a *plumb* line?

6. Define *vertical, horizontal,* and *inclined* lines, and give examples.

7. Define vertical, horizontal, and inclined planes, and give examples.

8. How can you test whether a line or a plane is vertical?

9. How can you test whether a line or a plane is horizontal?

10. When is a line said to *rotate* about a point? When does the rotation amount to one *revolution?*
11. Define a *circumference;* its *centre;* an *arc;* a *radius;* a *circle.*
12. What is true of radii of the same circle?
13. Describe how circles (arcs, etc.) are drawn.
14. In considering the directions of two lines, what two cases arise?
15. Define *parallel* lines, and give examples.
16. Show that parallel lines can never meet.
17. Are lines which would never meet always parallel? Give an example.
18. Give examples of parallel *curves;* of parallel *planes;* of a *line* and a *plane* parallel to each other.
19. How are parallel lines drawn?
20. What four cases arise when we consider the directions of three lines? Represent the four cases by figures.
21. How can we illustrate the truth that a straight line is the shortest distance between two points? Give an application of the same truth.
22. Define *equal* and *unequal* lines.
23. Explain how the equality and the inequality of two lines are expressed.
24. How is the equality of two lines tested?
25. What is an *Axiom?* Give Axioms I., II., III., IV. and V.
26. Explain how lines are added, subtracted, multiplied, and divided.
27. In what two senses of the word can *division* be performed on a line?
28. Define a *measure* of a line; a *multiple* of a line; a *common measure* of two or more lines; the *ratio* of two numbers and of two lines. What are the terms of a ratio called?
29. In what two ways can a ratio be expressed or written?
30. How is the ratio of two lines found?
31. What are *incommensurable* lines?
32. How can we approximate to the value of their ratio?
33. Define a *unit* of length; *measuring* a line; the *numerical value* and the *length* of a line.
34. What is the ratio of a foot to an inch? of a yard to a foot? of a rod to a yard? of a mile to a foot? What are the corresponding *reciprocal* ratios?
35. Define the *meter.*
36. Name and define the multiples and divisions of the meter.
37. How are metric units reduced from one denomination to another?
38. Which of the metric units are most used?
39. In expressing a length, how many units should be used?
40. How would you read a length containing a fraction of a unit?

41. In adding, subtracting, etc., metric units, what rule must be observed?

42. What is the value of one meter in inches?

43. How are lines on paper usually measured? longer lines, such as the length of a room? lines on the ground?

44. What are the advantages and disadvantages of *pacing* as a means of measuring a line?

45. Give examples of lines which cannot be directly measured.

46. What is a *reducing scale*, and what is *drawing to scale?*

47. How are scales of reduction named and written?

48. Why must the ratio or scale of reduction be constant for all lines belonging to the same map, plan, or diagram?

EXERCISES.

1. How many straight lines can be drawn through the North and South poles of the earth?

2. How many straight lines can be drawn through n points (n being any number), each line connecting two points?

Solution.—From the first point we can draw lines through all the other points; therefore, in all, $n-1$ lines. The same number may be drawn from each of the other points, and since there are n points, this would make $n(n-1)$ lines in all. But, in this way, two lines would be drawn in every case through the same two points; these two lines, however, would coincide and form but one line by § **22**, II. Therefore, the total number of different straight lines which can be drawn is $\frac{n(n-1)}{2}$.

3. Find the greatest possible number of straight lines which can be drawn through ten points. *Ans.* 45.

4. Through a vertical line how many vertical planes can be passed in thought (that is, how many different vertical planes can be imagined, all of which shall contain the vertical line)? how many horizontal planes? how many inclined planes?

5. Through a horizontal line how many vertical planes can be passed? how many horizontal planes? how many inclined planes?

6. Through an inclined line how many vertical planes can be passed? how many horizontal planes? how many inclined planes?

7. How many centres can a circle have? how many circles can have the same centre?

8. Can two curved surfaces be parallel to each other? a curved surface and a plane surface?

9. Can you draw a line parallel to a given line with the aid only of the dividers?

10. Find in how many points six lines can intersect one another.

11. Can you illustrate with numbers Axioms II.–V.?

12. If you multiply the ratio of two lines by their *reciprocal* ratio, what is the product always equal to?

13. Draw any two lines, and find their ratio.

14. Choose two lines in the room; estimate their ratio by the eye, then find it by measuring them.

15. $18^{m} + 4^{dm} - 64^{cm} + 268^{mm} =$ how many meters?

16. $9^{km} - 220^{m} + 84^{dkm} - 72^{hm} =$ how many kilometers?

17. The sum of two lines is 24 feet, and their difference is 9 feet; find the lines.

18. Two lines are to each other as 5 : 3, and their sum is 48^{m}; find the lines.

Ans. $\frac{5}{8} \times 48^{m} = 30^{m}$, and $\frac{3}{8} \times 48^{m} = 18^{m}$.

19. The sum of two lines is 84^{cm}, and their ratio is 5 : 16; find the lines.

20. The difference of two lines is 20^{m}, and their ratio is 4 : 9; find the lines.

Ans. $\frac{4}{5} \times 20^{m} = 16^{m}$, and $\frac{9}{5} \times 20^{m} = 36^{m}$.

21. I read in a newspaper that "the river Theiss has fallen 30^{cm}"; how much is this in inches?

22. Among the scales prescribed for the United-States engineer service are: general plans of buildings, 1 inch to 10 feet; maps $1\frac{1}{2}$ miles square, 2 feet to 1 mile; do., 3 miles square, 1 foot to 1 mile; do., from 4 to 8 miles square, 6 inches to 1 mile; do., 9 miles square, 4 inches to 1 mile; do., not exceeding 24 miles square, 2 inches to 1 mile; do., 50 miles square, 1 inch to 1 mile. Express these scales as fractions, with unity for the numerator.

Ans. $\frac{1}{120}, \frac{1}{2640}, \frac{1}{5280}, \frac{1}{10560}, \frac{1}{15840}, \frac{1}{31680}, \frac{1}{63360}$.

23. Which scale reduces lines the most: a scale of 1 inch to 1 mile, or ~~a scale~~ of 1 centimeter to 1 kilometer?

24. If the scale to which a map is made is not known, how can it be found?

CHAPTER III.

ANGLES.

I.—Definition of an Angle.

§ **44.** When we open the legs of the compasses (*Fig. 37*) they point in different directions: they are now said to make an ANGLE with each other. Replace, in thought, the legs by geometrical lines AB and AC, and we have an angle as understood in Geometry.

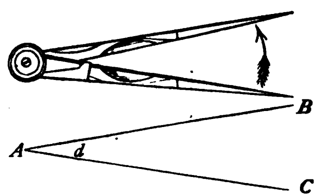

Fig. 37.

Definition. — *An* ANGLE *is the difference in direction of two lines.*

The two lines are called the SIDES of the angle, and the point where they meet is called the VERTEX.[1] If the lines do not actually meet (see *Fig. 25*, p. 27), the point where they would meet if prolonged, is the vertex of the angle.

An angle is *named* in one of three ways: —

(1) By a *small* letter placed just inside the vertex;

(2) By a *large* letter placed just outside the vertex;

(3) By three letters, one at the vertex, the others on the sides.

---

[1] Plural, vertices.

In the last case, in reading or writing the angle, the letter at the vertex must stand between the others.

For example: the angle of the lines AB and AC (*Fig. 37*) is called either *the angle d*, or *the angle A*, or *the angle B A C* or *C A B*. In general, the word *angle* need not be written; the letters *d*, or *A*, or *B A C*, or *C A B* are sufficient.

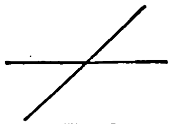

Fig. 38.

Exercises. — 1. Name the angles drawn on the blackboard.

2. Draw four angles; name them in different ways.

3. If (as in *Fig. 38*) several angles have the same vertex, which method of naming an angle cannot be used? Why not?

II.—Magnitude of an Angle.

§ **45.** Angles differ in magnitude. We shall get a clearer notion of the magnitude of an angle if we regard the angle as *generated or described by the rotation of a line.*

Hold one leg of the compasses at rest, and make the other rotate about the pivot in the direction shown by the arrow (*Fig. 37*); then the angle made by the leg will increase in magnitude. On the other hand, if we rotate the legs in the opposite direction, the angle will diminish; and when the legs are brought together, it will vanish entirely.

In like manner, *we may regard the angle made by any two lines, A B and A C* (*Fig. 37*), *as described by the rotation of one of the lines about their intersection A, until it coincides in direction with the other line.*

It is obvious that the *size* or *magnitude* of an angle depends entirely on *the amount of rotation* required to describe it, and not at all on the lengths of its sides.

Definition.— *Two angles are* EQUAL *if they can be placed one upon the other so that their vertices coincide in position, and their sides coincide in direction; otherwise the angles are* UNEQUAL.

We shall soon see that the equality of two angles (like that of two lines) can be tested without actually placing one upon the other.

Exercises. — **1.** Of the two angles on the blackboard, which has the longer sides? which is the greater angle?

2. Must the two sides of an angle have the same length?

§ **46.** If, in the angle BAC (*Fig. 39*), the side AC is made to rotate about the vertex A *away from* AB, until it comes into the position AD, it is clear that the angle BAD is equal to the *sum* of the angles BAC and CAD: that is, —

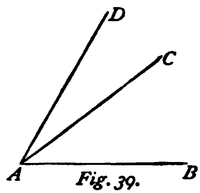

Fig. 39.

$$BAD = BAC + CAD.$$

If, on the other hand, the side AD of the angle BAD be made to rotate *towards* AB until it arrives at the position AC, there remains the angle BAC, which is equal to the *difference* of the angles BAD and CAD: in other words, —

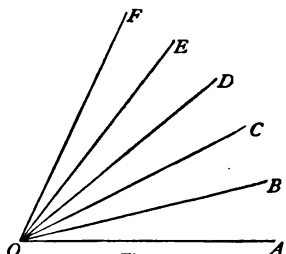

Fig. 40.

$$BAC = BAD - CAD.$$

If the angles AOB, BOC, COD, DOE, EOF (*Fig. 40*), are equal to one another, then $AOC = 2\,AOB$, $AOD = 3\,AOB$, $AOE = 4\,AOB$, $AOF = 5\,AOB$.

Conversely, it is evident that, —

$$AOB = \tfrac{1}{2}AOC = \tfrac{1}{3}AOD = \tfrac{1}{4}AOE = \tfrac{1}{5}AOF.$$

Angles, in the same sense as lines, may be added, subtracted, multiplied, and divided.

Definitions. — I. *To* BISECT *an angle (or a line) is to divide it into two equal parts.*

II. *A line which bisects an angle is called a* BISECTOR.

Exercises.—1. In *Fig. 40* what angle is equal to the sum $BOC + COE$? to the difference $AOF - COF$?

2. Draw freehand, three angles, of which one is twice, the other five times as large as the first.

3. Bisect freehand, a given angle.

4. How many bisectors can an angle have?

5. Divide an angle, freehand, into three, four, five, six equal parts.

6. In *Fig. 40* what is the ratio of the angles AOB and AOF? AOC and AOE?

III.—Magnitudes of Particular Angles.

§ **47.** Angles receive different names according to their magnitude.

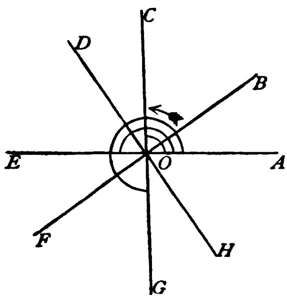

Fig. 41.

If we conceive a line to revolve round O (*Fig. 41*), starting from the initial position OA and revolving in the direction of the arrow, it will describe an angle of constantly increasing magnitude.

When the line has made exactly *one-fourth* of a complete revolution, the angle described, AOC, is called a Right Angle; it is sometimes denoted by the letter R.

When the line has made *one-half* of a revolution, it is in the position OE; the angle described, AOE, is equal to *two* right angles, and is called a Straight Angle, because its sides OA and OE, having opposite directions, form one straight line.

When the line has made *three-fourths* of a revolution, it is in the position OG. The angle which has been described, AOG, is equal to *three* right angles.

Finally, when the line has returned to the initial position OA, it has made a *complete* revolution, and has described *four* right angles.

Definitions.—I. A STRAIGHT ANGLE *is the angle described by a line which makes one-half of a revolution.*

II. *An angle less than a straight angle is called a* CONCAVE ANGLE; *an angle greater than a straight angle is called a* CONVEX ANGLE.

III. A RIGHT ANGLE *is one-half of a straight angle.*

IV. *A concave angle, if less than a right angle, is called an* ACUTE ANGLE; *if greater, it is called an* OBTUSE ANGLE.

In *Fig. 41*, $A\,O\,B$, $A\,O\,C$, $A\,O\,D$ are concave angles; $A\,O\,F$, $A\,O\,G$, $A\,O\,H$ are convex angles; $A\,O\,B$ is an acute, $A\,O\,D$ an obtuse angle.

NOTES.—1. Acute and obtuse angles are sometimes called *oblique* angles, in distinction from the right angle.

2. Whenever two lines meet, there is formed both a concave and a convex angle; but, when not otherwise stated, the concave angle is to be understood.

Since the sides of a straight angle form a straight line, and one straight line can always be placed on another straight line so as to coincide with it in direction, it follows that two straight angles satisfy the test of equal angles (§ **45**); in other words,—

All straight angles are equal to one another.

And since a right angle is half of a straight angle, it follows (by Axiom III.) that—

All right angles are equal to one another.

Exercises.—**1.** Open the legs of the compasses so that they make a right angle; a straight angle; three right angles; an acute angle; an obtuse angle; a concave angle; a convex angle.

2. On the blackboard are several angles: what kind is each, and why?

3. What kind of angles are found on the cube? the prism? the pyramid?

4. Point out angles on various objects (doors, windows, tables, walls of the room, etc.), and name the kind in each case. What kind of angle occurs most frequently?

5. Draw, free-hand, a right angle; an acute angle; an obtuse angle; a concave angle; a convex angle.

6. At 12 o'clock what angle do the hands of a watch make with one another? at 3 o'clock? at 6 o'clock? at 9 o'clock.

7. Mention a time of the day when the angle between the hands of a watch is an acute angle? a right angle? an obtuse angle?

8. What part of one revolution does the minute-hand describe in 5 minutes? (*Ans.* $\frac{5}{60}=\frac{1}{12}$.) In 10, 20, 30, 40, 50 minutes?

9. How long does it take the minute-hand to describe a right angle? the hour-hand?

10. What *kind* of an angle does the hour-hand describe in 1 hour? in 3 hours? in 5 hours? in 6 hours? in 8 hours? in 9 hours? in 12 hours?

11. What kind of an angle does a vertical line make with a horizontal line?

12. Draw a right angle with one side vertical; with one side horizontal; with one side inclined.

§ 48. Definitions.—I. *A straight line is said to be* PERPENDICULAR *to another straight line when it makes a right angle with it.*

II. *Two lines not perpendicular to each other are said to be* INCLINED *to each other.*

Examples.—The lines AE and CG (*Fig. 41*), the sign + in arithmetic, a common cross, the slats of a window (*Fig. 42*), are examples of perpendicular lines; the lines AB and AC (*Fig. 39*) are inclined lines.

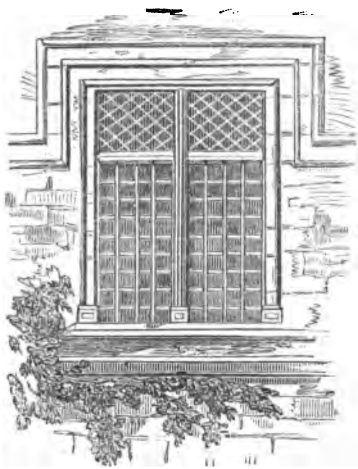

Fig. 42.

The sign ⊥ is sometimes used for the word *perpendicular;* thus: $AE \perp CG$ is read: *the line AE is perpendicular to the line CG.*

Exercises.—1. Give examples of perpendicular lines; of inclined lines.

2. What is the difference between a *perpendicular* line and a *vertical* line?

3. If two lines are perpendicular to each other, and one of them is vertical, must the other be horizontal?

4. If two lines are perpendicular to each other, and one of them is horizontal, must the other be vertical?

5. Give an example of two perpendiculars, one of which is vertical, the other horizontal.

§ **49.** Among the most common problems which the surveyor, carpenter, draughtsman, etc., have to solve, are —

(*i.*) To *erect* a perpendicular at a given point of a given line.

(*ii.*) To *let fall* a perpendicular from a given point to a given line.

On the ground, surveyors use various instruments, the simplest of which is the *Surveyors' Cross* (*Fig. 43*).

It consists of a block of wood having two saw-cuts made very precisely at right angles to each other. This block is fixed on a pointed staff on which it can turn freely. To *erect* a perpendicular, set the cross at the point of a line where a perpendicular is wanted, then turn the block till on looking through one saw-cut you see the ends of the line; then the other saw-cut will point out the direction of the perpendicular. To *let fall* a perpendicular to a line from some object (as the corner of a field, a tree, etc.), set the cross at a point of the line which seems to the eye about right, then note how far from the object the perpendicular at this point strikes, and move the cross that distance; repeat this operation till the correct point is found.

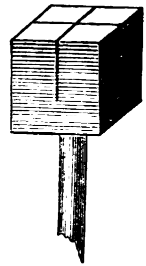

Fig. 43.

On paper, or the blackboard, both of the above problems are readily solved by means of the *Ruler* and the *Square.* Explain how.

NOTE.— The method of solving these problems on paper with the compasses will be shown in Chapter IV.

Exercises.—1. Draw a straight line, and then, free-hand, (*a*) erect a perpendicular at a point on the line, (*b*) let fall a perpendicular from a point not on the line. Test and correct the results by the aid of the ruler and the square.

2. How many perpendiculars can be erected at a given point of a given line? How many can be let fall from a given point to a given line?

3. Draw a horizontal line, and, at five points on it, erect perpendiculars. What kind of lines are they?

4. Draw a vertical line, and, at five points on it, erect perpendiculars. What kind of lines are they?

5. Draw an inclined line, and, at five points on it, erect perpendiculars. What kind of lines are they?

6. Draw a horizontal line, erect a perpendicular, and at a point of this perpendicular erect also a perpendicular. What direction has the second perpendicular?

7. Erect a perpendicular at a point of a line on the ground.

8. Choose a point in the yard (playground or field), and let fall a perpendicular to the nearest side of the yard.

IV.—Measure of an Angle.

§ **50.** Angles (like lines) are measured by choosing a *unit*, and then comparing other angles with this unit. The unit for measuring angles is obtained by dividing the right angle (which would be too large a unit) into ninety equal parts, and calling one of these parts a *degree*. The degree is subdivided into *minutes* and *seconds*.

One ninetieth part of a right angle is called a DEGREE.

One sixtieth part of a degree is called a MINUTE.

One sixtieth part of a minute is called a SECOND.

Abbreviations.—Degree = $^\circ$; minute = $'$; second = $''$.

Thus, $24^\circ\ 17'\ 8''$ is read: 24 degrees 17 minutes 8 seconds.

The values in degrees of the angles defined in § **47** are as follows:—

A right angle	$= 90^\circ$.	An acute angle	$< 90^\circ$.
A straight angle	$= 180^\circ$.	An obtuse angle	$> 90^\circ$.
Three right angles	$= 270^\circ$.	A concave angle	$< 180^\circ$.
Four right angles	$= 360^\circ$.	A convex angle	$> 180^\circ$.

Exercises.—**1.** How many seconds in 1°? in 10°? in 45°? in 360°?

2. Reduce $48^\circ\ 54'\ 36''$ to seconds.

3. Reduce $120000''$ to degrees.

4. Add $37^\circ\ 48'\ 35''$, $28^\circ\ 39'$, and $78^\circ\ 9'\ 55''$.
5. From $128^\circ\ 15'\ 31''$ take $69^\circ\ 42'\ 18''$.
6. Multiply $18^\circ\ 35'$ by 2, 3, 4, and 5.
7. Divide $72^\circ\ 27'$ by 2, 3, 4, and 5.
8. Reduce to degrees $5\,R$ (5 right angles); $6\,R$; $8\,R$; $12\,R$; $\frac{1}{2}\,R$; $\frac{3}{2}\,R$; $\frac{5}{2}\,R$; $\frac{1}{3}\,R$; $\frac{1}{4}\,R$; $\frac{1}{5}\,R$; $\frac{1}{6}\,R$; $\frac{1}{8}\,R$; $\frac{1}{10}\,R$; $\frac{1}{12}\,R$.
9. How many right angles in 540°? 900°? 225°? 30°? 5°?
10. Find the ratio between 10° and three right angles.

§ **51.** In measuring angles we take advantage of a simple relation between angles and the *arcs* intercepted between their sides, and described with the same radius from their vertices as centres.

In *Fig. 44* compare the angles AOB, AOC, AOD, with the arcs AB, AC, AD, intercepted by their sides. We see that the greater the angle the greater the *corresponding* arc, or arc intercepted by its sides. This conclusion is general: *in the same circle, the greater the angle at the centre the greater the corresponding arc, and the less the angle the less the arc.*

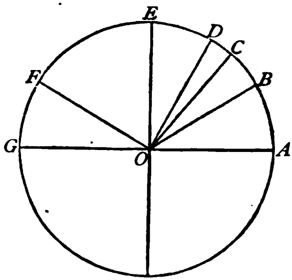

Fig. 44.

Again, if the two angles AOB and FOG are equal, the corresponding arcs AB and FG are also equal. For the angles, being equal, may be placed one on the other so that they will coincide; and then the arcs AB and FG must also coincide, since all the points of one arc are at the same distance from the centre O as the points of the other arc.

In like manner it can be shown that if the arcs AB and FG are equal, the corresponding angles at the centre, AOB and FOG, will also be equal.

The same reasoning may be applied with the same result to any angles and their corresponding arcs; hence, in general:—

I.—*Equal angles at the centre of a circle intercept on the circumference equal arcs.*

II.—*Equal arcs on the circumference correspond to equal angles at the centre.*

Accordingly, the circumference of a circle is divided into 360 equal arcs, each corresponding to an angle of $1°$ at the centre. These arcs are also called *degrees*, and are subdivided like the angular degree into *minutes* and *seconds*. And, in practice, an angle is measured by finding how many degrees, minutes, and seconds there are in the corresponding arc, it being obvious that *the number of angular degrees, minutes, and seconds in an angle is the same as the number of arc degrees, minutes, and seconds in the arc described with any radius from the vertex of the angle as a centre, and intercepted between its sides.*

We have just said that the arc employed to measure an angle may be described with *any* radius. With a longer radius we have, it is true, a longer arc (*Fig. 45*), but each division of the arc is also correspondingly increased in length, so that the *number* of divisions (degrees, etc.) in the entire arc remains the same as before.

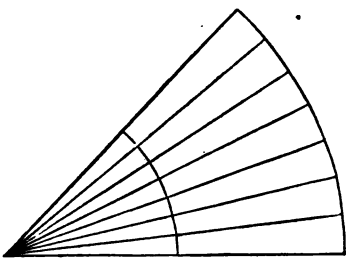

Fig. 45.

We are now prepared to answer the question: How can the equality of two angles be tested without placing one of them upon the other? We reply: two angles are equal, if, when their corresponding arcs are measured (as shown in the next section), these arcs are found to contain the same number of degrees, minutes, and seconds.

NOTE.—How an arc can be divided into degrees, etc., is a question which must be deferred until we know more about the properties of the circle.

Exercises.—**1.** Find the angle described by the hour-hand of a watch in 1 hour; 2 hours; 5 hours; 12 hours.

2. Find the angle described by the minute-hand of a watch in 1, 5, 10, 15, 20, 30, 45 minutes.

3. In how many minutes does the minute-hand describe an angle of 1°? 60°? 225°? 300°?

4. What angle do the hands of a watch make with each other at 1 o'clock? at 2, 3, etc., up to 12 o'clock?

5. The earth turns on its axis in 24 hours; what angle does a point on the surface describe in 1 hour? in 6, 12, 15 hours? 52 hours?

§ **52.** On paper or the blackboard, when great accuracy is not required, angles are measured with an instrument called a PROTRACTOR (*Fig. 46*). It is a semicircle, made of paper, horn,

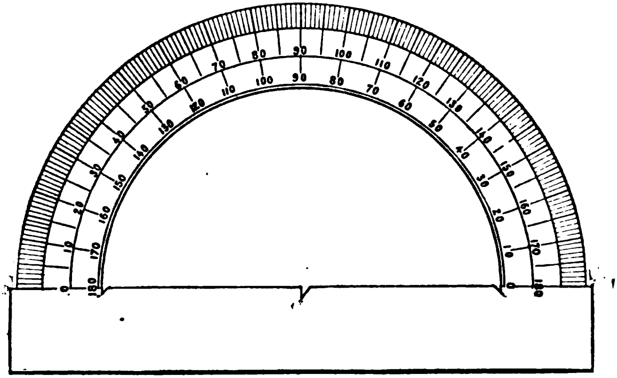

Fig. 46.

brass, or silver, the circular edge of which is divided into 180 equal parts or degrees. To measure an angle with the protractor, place the centre of the instrument on the vertex of the angle, and its zero line on one side; then read off on the edge of the protractor the division through which the second side of the angle passes.

On the ground, surveyors and engineers employ for measuring

angles, costly instruments called THEODOLITES. A cheap substitute for a theodolite is shown in *Fig. 47.* It consists of two pieces of wood shaped like rulers mounted on a vertical axis, by a pin driven

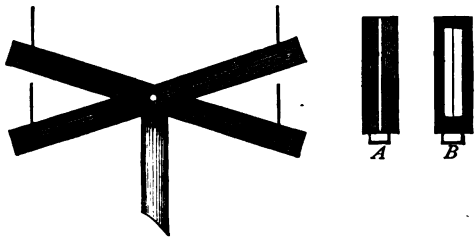

Fig. 47.

through their exact centres. The vertical needles inserted near the end of the rulers are used for *sighting.* In place of the needle nearest the eye, it is better to employ a thin strip of wood, *A*, having a fine vertical slit; and in place of the other needle, a vertical wire fixed in a light frame, *B*.

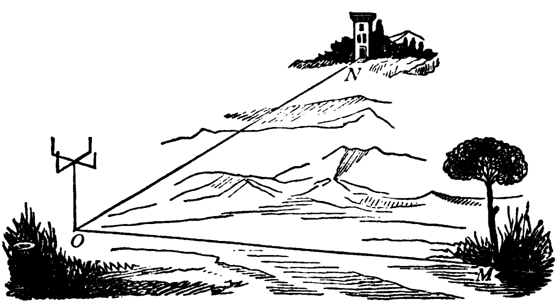

Fig. 48.

By the help of this instrument, and a protractor, one can measure with considerable accuracy an angle on the ground; for instance, the angle MON (*Fig. 48*).

Exercises.—1. On the blackboard are several angles. Measure each with the protractor, and write the result between its sides.

2. Draw five different angles. Estimate the value of the first in degrees; then measure it with a protractor. Proceed in like manner with each of the other angles. Give the results in 3 columns: in column 1, the estimated values; in column 2, the measured values; in column 3, the differences between the estimated and the measured values.

3. From a point on a straight line draw two lines, both on the same side of the given line; measure each of the three angles thus formed, and add the results. What is the sum? What ought the sum to be?

4. Through a point draw three lines; measure the six angles thus formed about the point; add the results. What is the sum? What ought it to be?

5. Open the legs of the compasses so that they make angles of $90°$; $60°$; $45°$; $30°$.

6. Imagine two lines drawn from your eye, one to the right-hand upper corner of the room, the other to the left-hand upper corner; what angle would the lines make?

7. With the aid of the protractor make an angle equal to a given angle?

8. Draw angles equal to $20°$; $30°$; $40°$; $60°$; $90°$; $100°$; $150°$; $180°$; $15°$; $45°$; $79°$; $81°$; $142°$.

9. Explain how the instrument in *Fig. 47* is constructed, and how you would proceed to measure by means of it the angle MON in *Fig. 48.*

V.—Angles made by Two Lines.

§ **53.** The angles AOC and COB (*Fig. 49*) are called *adjacent* angles.

Definition I.—*Two angles are* ADJACENT *when they have the vertex and one side common, and the other sides are opposite parts of the same straight line.*

We see from the figure that—

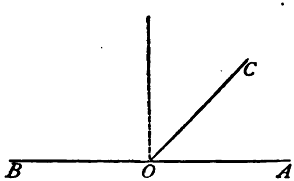

Fig. 49.

$$AOC + BOC = AOB = 2R.$$

Since this equation would hold true in whatever direction the line OC was drawn, we conclude that in *all* cases—

The sum of two adjacent angles is two right angles, or $180°$.

Definition II.—*If the sum of two angles is* 180°, *each is called the* SUPPLEMENT *of the other.*

Adjacent angles are always supplements of each other.

Exercises.—1. If one of two adjacent angles is a right angle, what is the other?

2. If one of two adjacent angles is acute, what is the other?

3. If one of two adjacent angles is obtuse, what is the other?

4. If one of two adjacent angles is known, how is the other found?

5. How is the supplement of a given angle found?

6. Find the adjacent angles of the following angles: 10°; 30°; 45°; 75°; 99°; 100°; 179°; 15° 48′; 79° 13′ 52″. What is the *supplement* of each of these angles?

§ **54.** Whenever two lines cross one another there are formed about the intersection of the lines four angles, a, b, c, d (*Fig. 50*). Of these angles the pairs a and b, b and c, c and d, d and a are adjacent angles; the pairs a and c, b and d are called *vertical* angles.

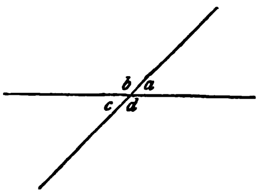

Fig. 50.

Definition.—*Two angles are called* VERTICAL *angles if they have a common vertex, and the sides of one angle have opposite directions to the sides of the other angle.*

If we measure two vertical angles, as a and c (*Fig. 50*), either with a protractor or by cutting them out on a piece of cardboard and laying them one upon the other, we shall find that they are equal. But without measuring them we can *prove* that they must be equal, if we bear in mind what has been said about adjacent angles in the last section. Since b is an adjacent angle to both a and c, therefore,—

$$a + b = 2R$$

and $$b + c = 2R.$$

Hence it follows from Axiom I. (state the axiom) that,—

$$a + b = b + c$$

subtract $\qquad b = b$

There remains $\qquad a = c$

For, if equals are taken from equals, the remainders are equal (Axiom III.).

Since this reasoning holds good, however the intersecting lines cut each other, we come to the general conclusion, that—

Two vertical angles are always equal to each other.

Remark.—In the above reasoning we have made use of the general truth shown in the last section; namely, that *the sum of two adjacent angles is* 180°; and also of Axioms I. and III. By reasoning upon these truths we have *proved* or *demonstrated* a new truth; namely, that *two vertical angles are equal.* Geometrical truths which are capable of proof by reasoning from known truths are called THEOREMS.

Exercises.—**1.** Draw adjacent and vertical angles and name them.

2. What is the sum of the angles a and b (*Fig. 50*)? Why?

3. What is the sum of the angles a, b, c, and d (*Fig. 50*)? Why?

4. One of the angles formed by two intersecting lines is 36°; find the others.

5. One of the angles formed by two intersecting lines is 90°; find the others.

6. Draw any two intersecting lines, and find the values of the angles formed at the intersection.

7. If one of the angles formed by two intersecting lines is known, how are the others found?

8. Each of two vertical angles is 45°; find the value of each angle in the other pair of vertical angles.

9. Prove that $b = d$ (*Fig. 50*) by reasoning like that used above to show that $a = c$.

VI.—*Angles made by Three Lines.*

§ **55.** Thus far the angles considered have had a common vertex; let us now consider angles formed about two different ver-

tices. Such angles are formed when two straight lines are cut by a third line.

The most important case, and the only one which we shall examine, is that in which two *parallel* lines, AB and CD (*Fig. 51*), are cut by a third line, EF. There are formed about the two points of intersection *eight* angles. These angles receive special names.

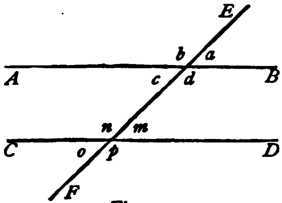

Fig. 51.

The four angles, c, d, n, m, which lie between the parallel lines, are called INTERNAL ANGLES; the other four, a, b, o, p, are called EXTERNAL ANGLES.

An external angle and an internal angle, as a and m, having different vertices, and lying on the *same* side of the intersecting line, are called CORRESPONDING ANGLES.[1]

An external angle and an internal angle, as a and n, having different vertices, and lying on *different* sides of the intersecting line, are called CONJUGATE ANGLES.

Two external angles, as a and p, or two internal angles, as d and m, having different vertices, and lying on the *same* side of the intersecting line, are called OPPOSITE ANGLES.

Two external angles, as a and o, or two internal angles, as d and n, having different vertices, and lying on *different* sides of the intersecting line, are called ALTERNATE ANGLES.

Exercises.—1. Name the four pairs of corresponding angles in *Fig. 51*; the four pairs of conjugate angles; the four pairs of opposite angles; the four pairs of alternate angles.

2. Which are *external*, which *internal* opposite angles? which are *external*, which *internal* alternate angles?

3. Which is the angle *corresponding* to o? *conjugate* to p? *opposite* to m? *alternate* to n?

---

[1] They are also called *external-internal* angles.

4. Draw two parallel lines, and a third line intersecting them. Letter the eight angles e, f, g, h, and p, r, s, t. Then write in six vertical columns (a) the external angles; (b) the internal angles; (c) the corresponding angles; (d) the conjugate angles; (e) the opposite angles; (f) the alternate angles.

§ **56.** If we conceive the line AB to move along the line EF, remaining always parallel to its first position, then, since its direction does not change, the four angles a, b, c, d which it makes with EF do not change. When AB reaches the parallel line CD, it will coincide with it in direction, and the four angles a, b, c, d will coincide respectively with the *corresponding* angles m, n, o, p. Moreover, each pair of *conjugate* angles, as a and n, become now *adjacent* angles; therefore their sum is 180° (§ **52**): the same is true of each pair of *opposite* angles, as a and p; lastly, each pair of *alternate* angles, as a and o, become now *vertical* angles, so that they are equal (§ **53**). Hence,—

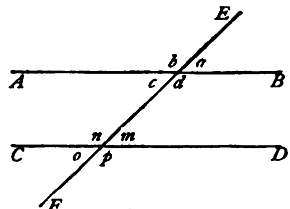

Fig. 52.

1.	2.	3.	4.
$a = m.$	$a + n = 2R.$	$a + p = 2R.$	$a = o.$
$b = n.$	$b + m = 2R.$	$b + o = 2R.$	$b = p.$
$c = o.$	$c + p = 2R.$	$c + n = 2R.$	$c = m.$
$d = p.$	$d + o = 2R.$	$d + m = 2R.$	$d = n.$

Putting these results into words, we obtain the following theorem:—

Theorem.—*If two parallel lines are cut by a third line,*—

1. The corresponding angles are equal;

2. The sum of two conjugate angles is two right angles;

3. The sum of two opposite angles is two right angles;

4. The alternate angles are equal.

Remark. — Theorems often involve consequences which are easily deduced from the theorems: such consequences are called in Geometry COROLLARIES. The present theorem furnishes an example.

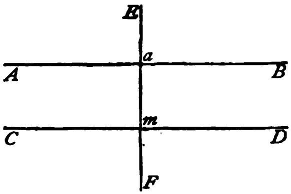

Fig. 53.

Corollary. — If $a = 90°$ (that is, if $AB \perp EF$), it follows that $m = 90°$ (that is, that $CD \perp EF$); or, to state the corollary in general terms, —

If a straight line is perpendicular to one of two parallels, it is also perpendicular to the other.

Exercises. — **1.** Give the proof separately for each of the last three equations in each of the above four series of equations.

Models. $b = n$, because when AB coincides with CD, b coincides with n.
$b + m = 2R$, because when AB coincides with CD, b becomes adjacent to m, and the sum of two adjacent angles is 180°.

2. Find the other seven angles when $a = 112°$; when $a = 90°$.

3. Draw two parallel lines, and a line intersecting them; then find the eight angles thus formed, using a protractor as little as possible.

4. If either one of the four parts in the theorem of this section is not true, what can we conclude as to the lines AB and CD?

§ **57. Theorem.** — *If two straight lines ~~are cut by~~ a third line so that two corresponding angles are equal, then the lines must be parallel.*

If, for example, $a = m$ (*Fig. 52*), then is $AB \parallel CD$. For, if we move AB towards CD, keeping it parallel to its first position, during the motion the angle a does not change because the direction of AB does not change. When the intersection of AB and EF coincides with the intersection of CD and EF, AB must coincide in direction with CD, since $a = m$. Therefore, AB, in its first position, must be parallel to CD.

Corollary. — By making in this theorem $a = m = 90°$, we obtain the following corollary: —

If two straight lines are both perpendicular to a third line, they are parallel. (*Fig. 54.*)

Remark.—All the four properties enumerated in the theorem of the last section are so related that if either one of them is true the other three must also be true, and the two intersected lines must be parallel.

Hence arises the importance of corresponding, conjugate, opposite, and alternate angles. Before we can assert with absolute certainty that two lines are parallel, we ought to show that the lines, when prolonged farther than even the imagination can reach, would not meet. Such an actual extension of the lines is of course impossible; it is, however, quite unnecessary, for the parallelism of two lines is very simply tested by the angles which are formed when the lines are cut by a third line.

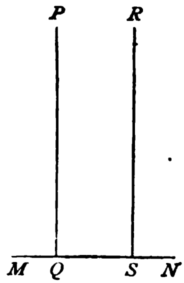

Fig. 54.

Exercises.—**1.** Prove the corollary, independently of the theorem, with lines drawn and lettered as in *Fig. 54*.

2. When more than two lines are perpendicular to a third, are they all parallel to each other?

§ **58.** In *Fig. 55*, we have drawn $AB \parallel CD$ and $EF \parallel GH$. If we compare the angle a with either of the angles x, y, z, we see that their sides are respectively parallel. But there are these differences: (*i.*) the sides of x have the *same* direction respectively as those of a; (*ii.*) the sides of z have *opposite* directions; (*iii.*) the sides of y have, one the *same* direction, the other the *opposite* direction.

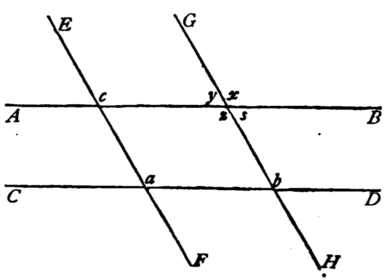

Fig. 55.

In case (*i.*), $a = b$ (corresponding angles on the parallels EF and GH), and $x = b$ (corresponding angles on the parallels CD and AB) : therefore $a = x$ (Axiom I.).

Theorem I. *If two angles have their sides respectively parallel, and directed the same way from the vertex, the angles are equal.*

In case (*ii.*), $x = z$ (vertical angles). And $a = x$ (Theorem I.). Therefore $a = z$ (Axiom I.).

Theorem II. *If two angles have their sides respectively parallel, and directed opposite ways from the vertex, the angles are equal.*

In case (*iii.*), $x + y = 2R$ (adjacent angles). But $a = x$ (Theorem I.). Therefore $a + y = 2R$.

Theorem III. *If two angles have their sides respectively parallel, and directed, the one pair the same way, the other pair opposite ways, from the vertex, the sum of the angles is two right angles.*

Exercises. — **1.** In *Fig. 55*, if $a = 115^\circ$, what is the value of x? of y? of z?

2. Prove Theorem I., using in the proof the angle c in *Fig. 55*, instead of the angle b.

3. What relation exists between a and s (*Fig. 55*)? Why?

4. Prove Theorems I., II., and III., with a new figure, drawn and lettered differently from *Fig. 55*.

REVIEW OF CHAPTER III.

QUESTIONS.

1. Define an *angle;* its *sides;* its *vertex.*
2. How is an angle named?
3. How may an angle be conceived as produced by motion?
4. On what does the magnitude of an angle depend?
5. Define *equal* angles.
6. Explain (by figures) what is meant by adding, subtracting, multiplying, and dividing angles.

7. What is *bisecting* an angle or a line? What is the *bisector* of an angle?
8. Define a *straight* angle; a *concave* angle; a *convex* angle; a *right* angle; an *acute* angle; an *obtuse* angle.
9. Show that all straight angles are equal; also, that all right angles are equal.
10. When are two lines *perpendicular* to each other? when are they *inclined?*
11. What are two of the commonest problems in surveying and drawing?
12. Describe the *surveyor's cross,* and how to use it.
13. How are perpendiculars to a given line drawn on paper?
14. Define a *degree,* a *minute,* and a *second.*
15. What are the values in degrees of the angles mentioned in Question 8?
16. Explain why an angle can be measured by means of the arc of a circle intercepted between its sides.
17. Why is the measure of an angle the same for any radius of the arc?
18. How is the equality of two angles tested without placing one of them on the other?
19. Describe the *protractor,* and how to use it.
20. Describe a simple substitute for a theodolite, and how an angle on the ground is measured by means of it.
21. Define *adjacent* angles; the *supplement* of an angle.
22. Show that the sum of two adjacent angles must be 180°.
23. Define *vertical* angles.
24. Two vertical angles are equal.[1]
25. What is a *theorem?*
26. Define *external* angles; *internal* angles; *corresponding* angles; *conjugate* angles; *opposite* angles; *alternate* angles.
27. If a straight line intersect two parallel lines, the corresponding angles are equal, the alternate angles are equal, and the sum of two conjugate angles, or of two opposite angles, is 180°.
28. What corollary follows from this theorem?
29. What is a *corollary?*
30. If two lines are cut by a third line so that two corresponding angles are equal, the two lines are parallel.
31. Give a corollary of this theorem.
32. How can you test if two straight lines are parallel?

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[1] In the review we shall always (as here) simply state the theorem to be proved or problem to be solved The learner is to understand that the proof (or solution) is required.

33. Two angles have their sides respectively parallel; then,—

(1) If these sides are directed the same way from the vertex, or if they are directed opposite ways, the angles are equal.

(2) If the sides are directed, the one pair the same way, the other pair opposite ways, the sum of the angles is 180°

EXERCISES.

1. What is the ratio of one right angle to three right angles? of one-third of a right angle to four right angles?

2. How can you test whether two intersecting lines are perpendicular to each other?

3. How can you test whether the two edges of a square are truly at right angles to each other?

Suggestion. — How was the straight edge of a ruler tested (§ **23**, Exercise 6)?

4. Make any angle, and then, by the aid of a protractor, make an angle four times as large; also, an angle one-fourth as large..

5. Mark three points, join them by lines, then measure the three angles which the lines make with one another. What is their sum?

6. Can you make three angles with two lines?

7. If two lines intersect, and one of the four angles is a right angle, prove that the other three are also right angles.

8. Draw five lines meeting at a point; how many angles are formed? what is their sum? If the angles are equal, find the value of each in degrees.

9. Ten lines meet at a point so as to form a regular ten-rayed star; find the value of the angle between any two rays.

10. What is the supplement of 1°? of 179°? of 180°?

11. Of two adjacent angles the greater is twice the less; find the values of the angles in degrees. What is the ratio of each to four right angles?

12. The bisectors of adjacent angles are at right angles to each other.

13. The bisector of one of two vertical angles bisects the other also.

14. If two parallel lines are cut by a third line, then (a) the bisectors of two alternate angles are parallel to each other; (b) the bisectors of two opposite angles are perpendicular to each other.

15. What is the difference between a *theorem* and an *axiom?* between a *theorem* and a *corollary?*

CHAPTER IV.

TRIANGLES.

CONTENTS.—I. Sides of a Triangle (§§ 59–63). II. Angles of a Triangle (§§ 64–66). III. Similarity, Equivalence, and Equality (§ 67, 68). IV. Equal Triangles (§§ 69–79). V. Some Consequences of the Equality of Triangles (§§ 80–93). VI. Applications (§§ 94–98).

I.—Sides of a Triangle.

§ **59.** In *Fig. 56* are several figures enclosed on all sides by lines. Each is a portion of a plane surface (the surface of the paper), and is called a *Plane Figure.*

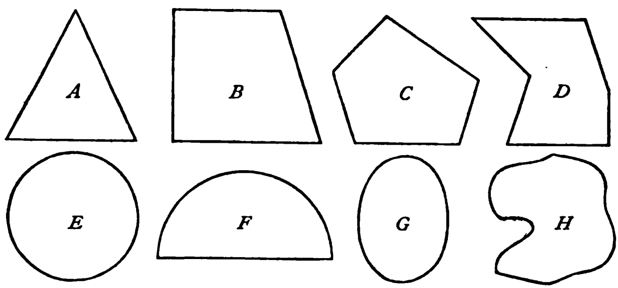


Fig. 56.

Definition I.—A PLANE FIGURE *is a portion of a plane surface bounded on all sides by lines.*

Many surfaces with which we are familiar,—for example, the outside of our houses, the floor and ceiling of a room, the leaf of a book, and, in many cases, gardens, fields, parks, etc.,—furnish in-

stances of plane figures bounded by *straight* lines. Such figures are called *Polygons.*

Definition II. — A POLYGON *is a plane figure bounded by straight lines.*

The bounding lines are called the SIDES of the polygon, and their sum is called the PERIMETER of the polygon.

Exercises. — **1.** In *Fig. 56* which figures are polygons and which are not polygons?

2. How many sides and how many angles has each polygon in *Fig. 56?*

3. Draw a polygon of five sides; six sides; eight sides; ten sides. How many angles has each?

§ **60.** Polygons receive different names according to the number of their sides.

At least *three* straight lines are required to enclose completely a part of a plane surface: the two sides of an angle are not sufficient. Therefore a polygon cannot have less than three sides.

Definition. — *A polygon of three sides is called a* TRIANGLE (*Fig. 57*).

A triangle is usually denoted by three letters standing at the vertices of its angles. Instead of writing the word *triangle*, the sign $\triangle$ is sometimes employed. Thus the triangle in *Fig. 57* is called the triangle $A B C$, or $\triangle A B C$.

Every triangle has SIX PARTS, three sides, $A B$, $A C$, $B C$ (*Fig. 57*), and three angles, A, B, and C.

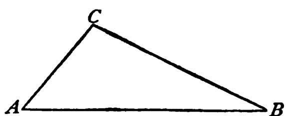


Fig. 57.

Each side, as $A B$, has two *adjacent* angles, A and B, and one *opposite* angle, C; each angle, as A, is *included* between two sides, $A B$ and $A C$, and is *opposite* to the third side, $B C$.

Exercises. — **1.** What angles are adjacent to the side $A C$ (*Fig. 57*)? to the side $B C$? What angle is opposite to $B C$?

2. By what sides is the angle B included? the angle C? What side is opposite to B? to C?

§ 61. If we prolong one side of a triangle, the prolongation makes with the adjoining side an angle called an EXTERIOR ANGLE of the triangle, and the three angles of the triangle are termed, by way of distinction, INTERIOR ANGLES.

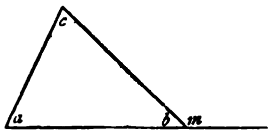


Fig. 58.

In *Fig. 58* m is an exterior angle, b is the *adjacent* interior angle; a and c are the *opposite* interior angles.

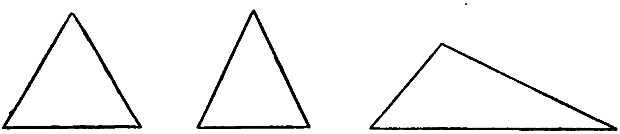
Exercises.—1. Draw a triangle, and then prolong each side in both directions. How many exterior angles are formed? For each exterior angle point out the adjacent interior angle, and also the opposite interior angles.

2. If an exterior angle of a triangle is 75°, find the adjacent interior angle.

§ 62. **Theorem.**—*The sum of two sides of a triangle is always greater than the third side.*

For the straight line AB (*Fig. 57*) is the shortest line from A to B (§ **32**), therefore less than the broken line ACB; that is, $AC + CB > AB$. For the same reason it follows that $AC + AB > CB$, and that $AB + BC > AC$.

Triangles are divided into three classes, according to the lengths of their sides: the EQUILATERAL (*Fig. 59*, I.), in which the three


I. II. III.
Fig. 59.

sides are equal, the ISOSCELES (*Fig. 59*, II.), in which two of the sides are equal, and the SCALENE (*Fig. 59*, III.), in which the sides are all unequal.

Exercises.—**1.** Can a triangle have for its sides 2^m, 3^m, and 6^m? 8^m, 4^m, and 2^m? 1^m, 2^m, and 3^m? 3^m, 5^m, and 7^m?

2. If the perimeter of a triangle is 2^m, what is the greatest value which one side can have?

3. Draw, free-hand (*a*), an equilateral; (*b*), an isosceles; (*c*), a scalene triangle.

4. The perimeter of an equilateral triangle is 1.5^m; find each side.

5. Can you draw accurately with the dividers an equilateral triangle of given side, say 60^{cm}?

§ **63.** We may regard a triangle as resting on one of its sides as a base; this side is then called the BASE of the triangle; the vertex of the angle opposite to the base is called the VERTEX of the triangle; and the perpendicular line drawn from the vertex to the base (or base prolonged) is called the ALTITUDE of the triangle. If the triangle has a horizontal side, this is usually taken for the base.

Thus in the triangle ABC (*Fig. 60*) we would naturally take the side AB for the base; then C is the vertex, and CD is the altitude. If AC is taken as the base, B is the vertex, and BE the altitude. If BC is taken as the base, A is the vertex, and AF the altitude.

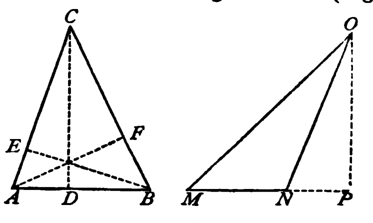


Fig. 60.

Notice that the three altitudes intersect in one point.

The altitude may be outside the triangle; this is the case in the triangle MNO, in which MN is the base, and OP the altitude. The altitude must fall outside if the triangle has an obtuse angle, and one of the sides of this angle is taken as the base.

In an isosceles triangle, the side unequal to the other sides is always taken as the base, and the equal sides are usually referred to as *the sides* of the triangle.

Exercises.—**1.** Draw a triangle having all its angles acute, and then draw the three altitudes.

2. Draw a triangle having an obtuse angle, and then draw the three altitudes. How many of the altitudes lie outside the triangle?

3. Draw an isosceles triangle and its altitude.

4. On a given line as base, how many isosceles triangles can be constructed? How many equilateral triangles?

II.—Angles of a Triangle.

§ **64.** In order to find the sum of the angles of a triangle ABC (*Fig. 61*), let us bring them all to a common vertex. For this purpose, suppose a line DE drawn through the vertex C of the triangle parallel to the base AB: we thereby make the angles m and n. The angles m and a are equal because they are alternate angles (§ **56**), and the angles n and b are equal for the same

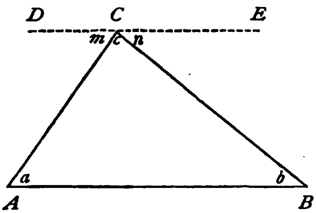


Fig. 61.

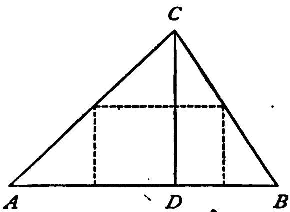


Fig. 62.

reason. Hence, $a + b + c = m + n + c$ (Axiom II.). But $m + n + c = 180°$, or a straight angle. Therefore, $a + b + c = 180°$ (Axiom I.).

This proof applies equally well to any triangle; therefore, to all triangles; hence:—

Theorem.—*The sum of the angles of a triangle is equal to two right angles, or* 180°.

Illustration.—The truth of this theorem may be shown to the eye as follows: Cut out of paper a triangle, ABC (*Fig. 62*), and draw its altitude CD. Then fold over the corners on the

dotted lines as edges. This will bring the vertices of the three angles to the same point D, and we shall *see* that the three angles together just make a straight angle.

In the same way we might test one triangle after another, finding in each case that the sum of the angles was 180°; after a time we should become convinced that all triangles were subject to this law.

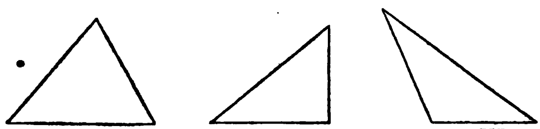
But in the *proof* given above a single case is sufficient; for a little reflection shows that the same reasoning holds good for all cases.

§ **65.** From the preceding important theorem several corollaries follow.

Corollaries. — 1. *The sum of two angles of a triangle must always be less than* 180°.

2. *A triangle can have only one right, or one obtuse, angle.*

In other words, two angles of a triangle must be acute; the third may be either acute, right, or obtuse. If it is acute, the triangle is called an ACUTE TRIANGLE (*Fig. 63*, I.); if it is right, the



I. II. III.

Fig. 63.

triangle is called a RIGHT TRIANGLE (*Fig. 63*, II.); if it is obtuse, the triangle is called an OBTUSE TRIANGLE (*Fig. 63*, III.).

In a right triangle the side opposite to the right angle is called the HYPOTENUSE, and the other sides are called the LEGS.

3. *If two angles of a triangle are known, the third angle will be found by subtracting the sum of the two known angles from* 180°.

4. *If two angles of one triangle are equal respectively to two angles of another triangle, then the third angle of the one is equal to the third angle of the other.*

5. *In a right triangle the sum of the acute angles is equal to a right angle or* 90°.

Hence, if one of the acute angles in a right triangle is known, the other can be found by subtracting the first from 90°.

Definition.—*If the sum of two angles is* 90°, *each is called the* COMPLEMENT *of the other.*

The acute angles of a right triangle are always complements of each other.

Exercises.—**1.** Can a triangle have one right *and* one obtuse angle?

2. If one leg of a right triangle is taken as base, what is the altitude?

3. Two angles of a triangle are,—

(*a*) 37° and 71°;	(*d*) 45° 32′ 18″ and 62° 50′ 57″;
(*b*) 82° and 48°;	(*e*) 64° 47′ 33″ and 77° 18′ 41″;
(*c*) 40° 28′ and 18° 57′;	(*f*) 179° 0′ 54″ and 0° 59′ 5″;

find the third angle in each case.

4. If one acute angle of a right triangle is (*a*) 30°; (*b*) 45°; (*c*) 63°; (*d*) 27° 15′; (*e*) 58° 12′ 48″, find in each case the other acute angle.

5. What is the complement of 30°? 45°? 60°? 89°? 90?

§ **66**. If we subtract the angle b (*Fig. 64*) from 180°, the remainder is equal to $a + c$ (§ **64**). But this remainder is also equal to the exterior angle, m; why (§ **53**)? Therefore, $m = a + c$ (Axiom I.). That is,—

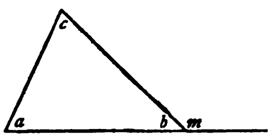


Fig. 64.

Theorem.—*An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

Exercises.—**1.** Find the exterior angle of a triangle if the opposite interior angles are 38° 35′ and 69° 46′.

2. An exterior angle of a triangle is 60°, and one of the opposite interior angles is 30°; find the other.

3. An exterior angle of a triangle is 90°. What kind of a triangle is it?

4. Prove that the sum of the six exterior angles of a triangle is 720°.

III. — Similarity, Equivalence, and Equality.

§ **67.** We may compare two triangles with respect either to their *shape* (*form*) or to their *size* (*magnitude*).

There are four cases. Either (*a*), the triangles have the same shape; or (*b*), they have the same size; or (*c*), they agree in both shape and size; or (*d*), they differ in both shape and size.

For example: in *Fig. 65* the triangles ABC and POQ have the same shape; ABC and DEF have the same size; ABC and MNO agree in both shape and size; DEF and POQ differ in both shape and size.

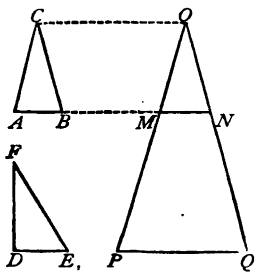


Fig. 65.

If two triangles have the same shape, they are called SIMILAR TRIANGLES; if they have the same size, they are called EQUIVALENT TRIANGLES; if they agree both in shape and size — that is, if they are both similar and equivalent — they are called EQUAL TRIANGLES.

Exercises. — 1. To which class of triangles do POQ and MON belong? MON and DEF?

2. Draw (free-hand) two triangles of each kind.

§ **68.** What has just been said about triangles may be extended to space magnitudes in general.

Two straight lines always have the same shape whatever be their lengths; so have two circles, two cubes, or two spheres, however much they differ in size.

Again, a curved line may have the same length as a straight line; a field bounded by curved lines may enclose the same extent of surface as a field bounded by straight lines; a cubical vessel may hold the same quantity of water as a vessel shaped like a cyl-

cylinder or a sphere. In all these cases the *size* is the same, the *shape* different.

Lastly, two magnitudes, whether lines, surfaces, or solids, may have at once the same size and the same shape.

Definitions. — I. *Two magnitudes which have the same shape are called* SIMILAR MAGNITUDES.

II. *Two magnitudes which have the same size are called* EQUIVALENT MAGNITUDES.

III. *Two magnitudes which agree both in shape and in size are called* EQUAL MAGNITUDES.

Between two similar magnitudes we place the sign $\sim$; between two equivalent magnitudes the sign $=$; between two equal magnitudes both signs $\cong$; or, if the shape is not taken into account, simply the sign $=$.

Of these three kinds of agreement between space magnitudes, that of equality is the simplest, and ought therefore to be studied first.

IV.—Equal Triangles.

§ **69**. Since two equal triangles agree both in size and in shape, they can differ from each other only in their *position* in space, and must, when one is laid on the other, coincide in all their six parts; in other words, their three sides and three angles must be equal each to each.

Compare, for example, the equal triangles ABC and MNO (*Fig. 65*) :—

Of the sides, $AB = MN$; $AC = MO$; $BC = NO$.

Of the angles, $C = O$; $B = N$; $A = M$.

In two equal triangles equal sides are opposite to equal angles, and equal angles are opposite to equal sides.

In any two triangles by *corresponding* sides (angles) are meant those which are opposite to equal angles (sides).

§ 70. There would be nothing more to say about equal triangles were it not for the fact that, when in two triangles a certain number of the six parts are equal each to each, the remaining parts *must* also be equal, and the triangles must be equal. This fact is one of the corner-stones in the science of Geometry; upon it are based the proofs of many important theorems, and it is likewise a fact of great practical value in the various uses to which Geometry can be applied. (See Part VI. of the present chapter.)

Let us therefore proceed to find, once for all, how many and what parts in two triangles must be equal in order to make the triangles equal; in other words, how many and what parts of a triangle must be given in order to *determine* the size and shape of the triangle.

1. If only one part, a side or an angle, is given, we can construct as many different triangles as we please, all having this part. Therefore, by one part the size and shape of a triangle is not determined.

2. Likewise, if two parts are given, whether two sides, two angles, a side and an adjacent angle, or a side and the opposite angle, it is easy to show that an indefinite number of triangles may be made, all having these two parts and yet differing in their other parts (see Exercises 3–6 below). Hence two parts are not sufficient to determine the size and shape of a triangle.

3. If three parts are given, they may be, —

(*i.*) The three angles;

(*ii.*) One side and two angles;

(*iii.*) Two sides and the included angle;

(*iv.*) Two sides and an angle opposite to one of them;

(*v.*) The three sides.

These cases require, each of them, a special investigation.

Exercises. — 1. Draw three triangles having a common side.

2. Draw three triangles having a common angle.

3. Draw two triangles, both having for two of their sides 40^{cm} and 70^{cm}.

4. Draw two triangles, both having for one side 50^{cm}, and for an adjacent angle 30°.

5. Draw three triangles, all having for one side 45^{cm}, and for the opposite angle 60°.

Solution.—Make an angle of 60°. Then take 45^{cm} between the dividers, place one point on one side of the angle, the other point on the other side. Join these two points and the triangle is constructed. The other triangles are made in the same way, taking care to use different points on the sides of the angle.

6. Draw two triangles, both having the angles 30° and 60°. What kind of triangles are they?

§ **71.** Is a triangle determined if we know its three angles? Compare the triangles ABC and DEF (*Fig. 66*); as regards their angles, $A = D, B = E, C = F$ (§ **58**); that is, the triangles have their angles equal each to each. Now the triangles themselves are not equal; they have the same *shape*, but they differ in *size*.

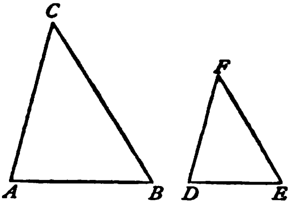


Fig. 66.

In fact we may make as many triangles as we please, all having the same shape, but differing in size; therefore,—

By the three angles a triangle is not determined.

Two triangles which have their angles equal each to each are said to be *mutually equiangular.*

Exercises.—1. Draw two mutually equiangular triangles.

2. Can you draw two mutually equiangular triangles which differ in shape?

§ **72. Problem.**—*To construct a triangle, having given a side and two angles.*

When two angles are given, the third can always be found (how?). We shall therefore suppose that the given angles are both adjacent to the side.

Let a (*Fig. 67*) be the given side, and 60° and 40° the given

angles. Draw a line $AB = a$. At the points A and B make angles equal respectively to 60° and 40°. The sides of these angles will meet at a point, C, and ABC is the triangle required.

Why cannot these sides be parallel, and hence never meet?

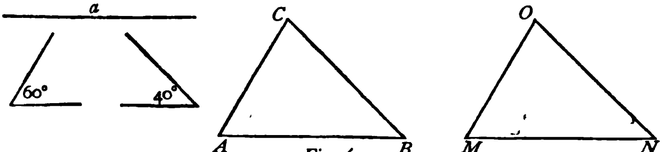


Fig. 67.

If we now construct another triangle, MNO, having the same three parts, it will be equal to the triangle ABC.

Proof. — Place MNO on ABC, so that MN falls on AB. M will fall on A, and N on B, since $MN = AB = a$. The lines MO and AC will coincide in direction, since the angles at M and A are equal (each 60°), and the lines NO and BC will also coincide in direction, since the angles at N and B are equal (each 40°). Therefore O must coincide with C, and the triangles will coincide in all their parts. Point out the parts which are equal each to each.

The construction and proof would be the same for other values of the given sides and angles; therefore, *by a side and two angles, a triangle is completely determined;* and also, —

Theorem (I. Law of Equality). — *If in two triangles a side and two angles are equal each to each, the triangles are equal.*

Remark. — The method used above to prove the two triangles equal is called the *Method of Superposition.* When it can be employed to prove the equality of two magnitudes, it is a very easy method, if we only take care to bear in mind what parts in the two magnitudes are known to be equal.

Exercises. — **1.** Draw to reduced scale (on paper, 1 : 2000; on the blackboard, 1 : 200) a triangle, one side of which is 220^m, and the adjacent angles 70° and 50°.

2. Draw to reduced scale the outlines of a triangular field of which one side is 600^{m}, and the adjacent angles 80° and 47°.

NOTE.—Always write the scale employed by the side of the figure.

3. The same Exercise, one side of the field being 40^{m}, an adjacent angle being 65°, and the *opposite* angle being 37°.

4. Draw a right triangle having given,—

(*i.*) One leg $= 8^{dm}$, and the adjacent acute angle $= 57^{\circ}$.

(*ii.*) One leg $= 60^{cm}$, and the opposite angle $= 25^{\circ}$.

(*iii.*) The hypotenuse $= 90^{cm}$, and an adjacent angle $= 42^{\circ}$.

5. What limit is there to the sum of the two given angles in the problem of this section in order that the solution may be possible?

6. What are the *given* things or *data*[1] in the problem of this section?

§ 73. Problem.—*To construct a triangle having given two sides and the included angle.*

Let a and b (*Fig. 68*) be the given sides, and 50° the included angle.

Make an angle $A = 50^{\circ}$, take $AB = a$, $AC = b$, and join BC. ABC is the triangle required.

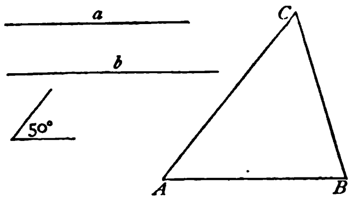


Fig. 68.

Draw another triangle having the three given parts, and then prove by superposition that it must be equal to ABC.

Hence, *by two sides and the included angle a triangle is completely determined;* and,—

Theorem (II. Law of Equality).—*If in two triangles two sides and the included angle are equal each to each, the triangles are equal.*

Exercises.—1. Draw (to reduced scale) a triangle with the sides 49^{m} and 77^{m}, and the included angle 45°.

2. Represent (to scale) on paper, and also on the blackboard, a triangle, two sides of which are 140^{m} and 100^{m}, and the included angle 55°. Are the two figures *equal, equivalent,* or *similar?* Are they mutually equiangular?

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[1] The singular of *data* is *datum.* The word is from the Latin, and means *given.*

3. In an isosceles triangle one of the equal sides is 32^{cm}, and the angle at the vertex is 82°; draw the triangle.

4. Draw a right triangle whose legs are 16^{cm} and 20^{cm}. What three parts are here known?

5. Draw an *isosceles* right triangle whose legs are 16^{cm} each. What are the three parts here given?

6. What is the greatest value which the given angle in the problem of this section can have?

7. What are the *data* in the problem of this section?

§ **74. Problem.** — *To construct a triangle having given two sides and an angle opposite to one of them.*

There are two cases according as the given angle is opposite (*i.*) to the greater, or (*ii.*) to the less side.

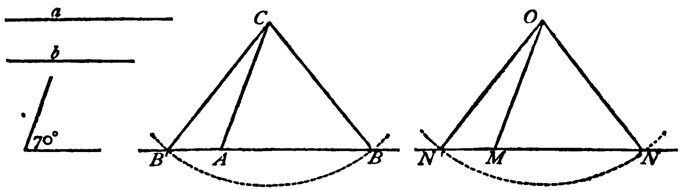

Fig. 69.

Case (*i.*). Let a and b (*Fig. 69*) be the given sides, and let $a > b$; also let the angle opposite to a be 70°. Make an angle $CAB = 70^{\circ}$, and lay off $AC = b$; this determines two corners, A and C, of the triangle. The third corner must lie in the line AB, and its distance from C must be equal to a; therefore, it must also lie in a circumference described from C as centre with a radius equal to a. Hence it must be the intersection of this circumference with the line AB. Now this circumference cuts AB in *two* points, B and B', so that we obtain *two* triangles, ABC and $AB'C$. Of these, however, only ABC has the three given parts; $AB'C$ has, it is true, the given sides, but not the given angle; therefore, it is not the triangle required.

If we construct another triangle, MNO, having the same three parts, it will be equal to the triangle ABC.

Proof. — Place $\triangle MNO$ on $\triangle ABC$, making the equal sides MO and AC coincide. MN will coincide in direction with AB because $OMN = CAB = 70°$; therefore N will fall in AB. But N must also lie in a circumference having (now) C for centre, and radius $= a$; therefore N will coincide with B. And ON will coincide with CB, and the triangles will coincide in all their parts. Hence we see that *by two sides and the angle opposite to the ~~greater side~~ a triangle is completely determined;* and that, —

Theorem (III. Law of Equality). — *If in two triangles two sides and the angle opposite to the greater side are equal each to each, the triangles are equal.*

Case (*ii.*). Let a and b (*Fig.* 70) be the given sides, and let $a < b$; also let the angle opposite to $a = 42°$.

By proceeding as in Case (*i.*) we obtain two triangles, ABC and $AB'C$, both having the three given parts, yet differing in size and in shape. Therefore, *by two sides and the angle opposite to the less side, a triangle is not determined.*

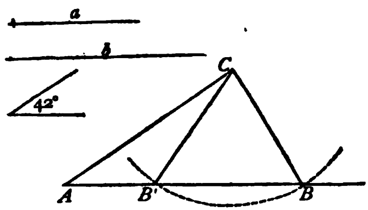

Fig. 70.

There are, however, *only two* solutions, the triangles ABC and $AB'C$.

Exercises. — **1.** In the triangles ABC and $AB'C$ (*Fig.* 69) point out the equal parts; also the unequal parts.

2. What relation exists between the angles BAC and $B'AC$ (*Fig.* 69)?

3. What kind of triangles are the triangles BCB' and NON' (*Fig.* 69)?

4. What equal parts have the triangles ACB and ACB' (*Fig.* 70)?

5. Draw a triangle with the sides 60^{cm} and 90^{cm}, and the angle opposite the greater side $76°$.

6. Construct a right triangle the hypotenuse of which is 8^{dm}, and one leg is 6^{dm}. What are the three given parts?

7. Construct a triangle with the sides 8^{dm} and 4^{dm}, and the angle opposite to the greater side 80°.

8. When are the data in Case (*ii.*) of this section such that no triangle is possible?

Ans. — When the less side is shorter than the perpendicular from C to $A B$ (*Fig. 70*); also when the given angle is either right or obtuse.

9. When the less side is just equal to the perpendicular from C to $A B$, what kind of a triangle is obtained?

§ **75. Problem.** — *To construct a triangle having given its three sides.*

Let a b c (*Fig. 71*) be the given sides. Make $A B = a$; then A and B are two corners of the triangle, and $A B$ is one side. Since the third corner must be at the distance b from A, it must lie in a circumference described from A as centre with the radius b. For a like reason it must also lie in a circumference described from B as centre with the radius c.[1]

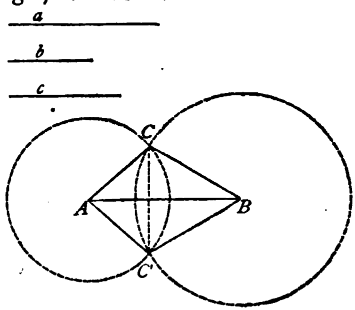

Fig. 71.

Therefore it must be at the intersection of these circumferences. Now these circumferences cut each other in *two* points, C and C', one above the other below $A B$; so that we obtain *two* triangles, $A B C$ and $A B C'$, both having the three given sides.

These triangles, however, are equal. For, if we fold over $A B C'$ on $A B$, C' will be brought *above* $A B$, and, as it remains the intersection of circumferences having A and B for centres, and b and c for radii, it will fall on C, and the triangles $A B C$ and $A B C'$ will then coincide.[2]

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[1] It is obvious that in solving this problem entire circumferences are not required; short arcs near the points of intersection are sufficient.

[2] See Exercise 5, page 92.

If we construct another triangle with the same three sides, it will also be equal to $A B C$. For we can place it on $A B C$ so that one of its sides will coincide with AB, and then its third corner must fall on C for the same reason that C' falls on C when ABC' is folded over on $A B$.

From what precedes we conclude that, *by the three sides a triangle is completely determined;* also that,—

Theorem (IV. Law of Equality).—*If in two triangles the three sides are equal each to each, the triangles are equal.*

Exercises.—**1.** The sides of a triangular garden measure 80^m, 65^m, and 45^m. Draw to scale a plan of the garden.

2. The sides of a triangular field are 200^m, 240^m, and 300^m. Draw to scale a plan of the field.

3. Construct a triangle whose sides are 20^{cm}, 40^{cm}, and 60^{cm}.

4. Construct a triangle whose sides are 20^{cm}, 30^{cm}, and 60^{cm}.

5. When is the solution of the problem of this section impossible? (See § **62**.)

6. Construct an isosceles triangle whose base is 70^{cm}, and the sum of whose other sides is 2^m.

7. Construct an equilateral triangle one side of which is 1^m.

8. Construct a triangle with the sides 30^{cm}, 30^{cm}, and 30^{cm}. What kind of a triangle is it?

9. Construct a triangle with the sides 25^{cm}, 25^{cm}, and 40^{cm}. What kind of a triangle is it?

10. Construct a triangle with the sides 20^{cm}, 30^{cm}, and 40^{cm}. What kind of a triangle is it?

11. Are two triangles equal if they have equal perimeters? (Compare the last three exercises.)

§ **76. Problem.**—*To make a triangle equal to a given triangle.*

Take any three parts of the given triangle that completely determine a triangle, and construct with them a new triangle; it will be equal to the given triangle. The best parts to take for this purpose are the three sides. Show how to construct the triangle with them (§ **75**).

Exercises. — **1.** In what four ways can three parts be chosen with which to construct a triangle?

2. Draw a triangle; then construct another equal to it.

§ 77. Problem. — *At a given point D (Fig. 72) of a given line D G, to make an angle equal to a given angle B A C.*

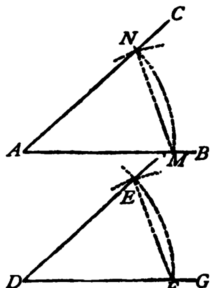


Fig. 72.

Take on the sides of the given angle $AM = AN$, and join MN. Then construct (by the last problem) a triangle DEF equal to the triangle AMN. The angle D, corresponding to the angle A, will be equal to it.

Exercises. — **1.** What kind of a triangle is AMN? Is it necessary for the solution of the problem to make AMN a triangle of this kind?

2. Make an angle, and then make another equal to it.

3. Make an angle equal to a given angle with the protractor.

4. Make an angle equal to a given angle with the ruler and the square.

5. Make (with ruler and compasses) an angle of 60°; an angle of 120°.

§ 78. Problem. — *Through a given point C (Fig. 73) to draw a line parallel to a given line A B.*

Through C draw any straight line CD, cutting AB in D. Then draw (by the last problem) CP so that the angle $DCP = CDB$. CP is parallel to AB (§ **57**).

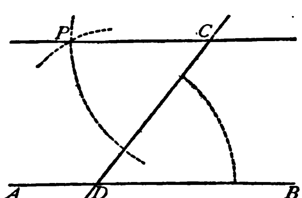


Fig. 73.

Exercises. — **1.** Draw a line, and then through a point draw another line parallel to the first.

2. The same exercise, using ruler and square instead of ruler and compasses.

3. Through the vertices of the angles of a triangle draw lines parallel to the opposite sides, and prolong them until they meet. What kind of a figure is thus formed?

§ **79.** A Problem is a construction to be made according to geometrical laws.

There is a general agreement that in the *Science* of Geometry no instrument shall be employed in making a construction except ruler and compasses; if the problem cannot be solved by these means it is not regarded as belonging to the science. In the *Art* of Geometry — that is, in the applications of Geometry to practical purposes, such as Drawing and Land-measuring — various other instruments which are found convenient are used, as divided rules, protractors, theodolites, etc.

Thus, in practice, the problem of § **77** is usually solved with the protractor, or with the ruler and the square, and the problem of § **78** with the ruler and the square.

In making a construction, all lines which are merely intermediate steps between the data and what is required should be drawn dotted instead of full (see *Figs. 72* and *73*). Such lines are called *auxiliary* lines.

V.—Some Consequences of the Equality of Triangles.

§ **80.** Let $A\,B\,C$ (*Fig. 74*) be an isosceles triangle having $A\,C = B\,C$, and let $C\,D$ be a line bisecting the angle C. This line divides the triangle $A\,B\,C$ into two triangles, $A\,C\,D$ and $B\,C\,D$. Compare these two triangles; they have a common side $C\,D$, the side $A\,C = B\,C$ (why?), and the angle $c = d$ (why?). Therefore the triangles are equal (II. Law of Equality), and among the equal parts we have $a = b$; hence,—

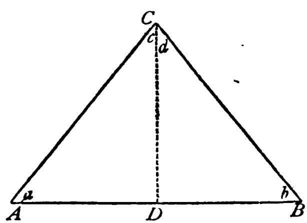


Fig. 74.

Theorem.—*In an isosceles triangle the angles opposite to the equal sides are equal.*

Corollary. — *In an equilateral triangle all the angles are equal; therefore each angle* $= 180° : 3 = 60°$.

Why does this corollary follow from the theorem?

Remark. — A theorem consists of three parts: —

(*i.*) The HYPOTHESIS (or hypotheses), or that which is given, or assumed as true, at the start.

(*ii.*) The CONCLUSION (or conclusions), or that which is to be proved.

(*iii.*) The PROOF.

In the above theorem the hypothesis is: $AC = BC$ (*Fig. 74*), and the conclusion is: $a = b$. The proof consists in showing that the triangles ACD and BCD are equal, and hence inferring that $a = b$.

Exercises. — **1.** In an isosceles triangle the bisector of the angle at the vertex is perpendicular to the base, and bisects the base.

Hints. — Use *Fig. 74*. The two hypotheses are: $AC = BC$, and CD *bisects* ACB. The two conclusions are: $CD \perp AB$, and CD *bisects* AB. Proof: show that $\triangle ACD \cong \triangle BCD$; hence $m = n = 90°$ (why?), and $AD = BD$.

2. The perpendicular let fall from the vertex of an isosceles triangle to the base bisects the base, and also the angle at the vertex.

Hints. — Here the hypotheses are: $AC = BC$, and $CD \perp AB$. The conclusions are: CD *bisects* AB, and CD *bisects* ACB. Prove by showing that $\triangle ACD \cong \triangle BCD$.

3. The line joining the vertex of an isosceles triangle to the middle of the base is perpendicular to the base, and bisects the angle at the vertex.

Hints. — In this case what are the hypotheses? the conclusions? Prove by showing (by IV. Law of Equality) that $\triangle ACD \cong \triangle BCD$.

4. What *three* conditions does the line CD (*Fig. 74*) fulfil?

5. Prove (with the aid of the theorem of this section) that the triangles ABC and ABC' (*Fig. 71*, p. 88) are equal.

Hint. — By drawing CC' we obtain two isosceles triangles, ACC' and BCC'.

6. Find the base angles of an isosceles triangle if the angle at the vertex is (*i.*) $27° \, 35'$, (*ii.*) $75° \, 18'$, (*iii.*) $124° \, 40'$.

7. Find the angle at the vertex if one of the base angles is (*i.*) $15° \, 14'$, (*ii.*) $48° \, 7'$, (*iii.*) $83° \, 4'$.

8. In an isosceles right triangle find the base angles.

9. Can you devise a way of finding the height of an object (tree, tower, etc.) by means of an isosceles right triangle?

10. Show that in an isosceles triangle the exterior angles made by prolonging the base are equal, and that the exterior angles at the vertex are also equal.

11. Find the three angles of an isosceles triangle if the *exterior* angle at the vertex is 146° 19′.

12. Find the three angles of an isosceles triangle if the exterior angle at the base is 120° 50′.

13. Show that the exterior angles of an equilateral triangle are all equal, and find the value of one of them.

14. Construct an isosceles triangle having given, —

(*a*) The base and an adjacent angle;
(*b*) The base and the opposite angle;
(*c*) The side and an angle at the base;
(*d*) The side and the angle at the vertex.

§ **81.** Let us now transpose the hypothesis and the conclusion of the theorem in § **80**; in other words, let us suppose that in a triangle *A B C* (*Fig. 75*) the angles *A* and *B* are equal, and inquire if the sides opposite to these angles must also be equal.

Let *CD* bisect the angle *C*. Then the triangles *A C D* and *B C D* are mutually equiangular (why?). They also have the side *C D* common, therefore they are equal (I. Law of Equality), and $AC = BC$. Hence, —

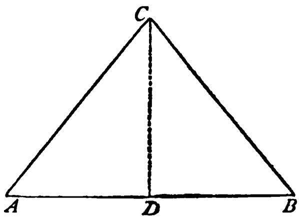


Fig. 75.

Theorem. — *If in a triangle two angles are equal, the opposite sides are also equal, and the triangle is isosceles.*

Corollary. — *If the three angles of a triangle are equal, the triangle is equilateral.*

Why does this corollary follow from the theorem?

Remark. — If we transpose the hypothesis and the conclusion

of a theorem, we obtain a new theorem called the CONVERSE of the original theorem. The theorem of this section, for example, is the converse of the theorem of the last section.

The converse of a true theorem is not necessarily true. Take, for instance, the theorem, *two vertical angles are always equal;* the converse, *two equal angles are always vertical angles*, is not true.

Exercises. — **1.** In an isosceles triangle the bisectors of the base angles form with the base another isosceles triangle.

2. Prove the converse theorem (namely, if in a triangle the bisectors of the base angles form with the base an isosceles triangle, then the triangle is also isosceles).

§ **82.** We will now compare in a triangle two unequal sides and the opposite angles.

Let ABC (*Fig. 76*) be an isosceles triangle having $AB = AC$ and $ABC = ACB$. Prolong AC to any point D, and join BD.

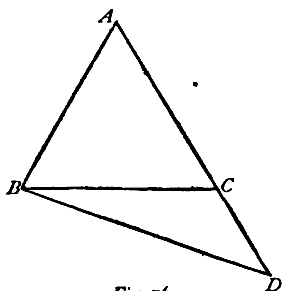


Fig. 76.

Then ABD is a triangle having $AD > AB$. Now compare the opposite angles. Since $ABD = ABC + CBD$, therefore $ABD > ABC$. And since $ADB = ACB - CBD$ (§ **66**), therefore $ADB < ACB$, and hence $< ABC$. Therefore $ABD > ADB$. Here it is quite clear that the two conditions, $AD > AB$, and $ABD > ADB$, must always be fulfilled together; hence, —

Theorem I. — *Of two angles of a triangle the greater is opposite to the greater side.*

Theorem II. (Converse of Theorem I.). — *Of two sides of a triangle the greater is opposite to the greater angle.*

Corollary. — *In a right triangle the hypothenuse is the greatest side.*

Exercises.—**1.** From which theorem does the corollary follow? Why does it follow?

2. What similar corollary follows for an obtuse triangle?

§ **83.** Let $CD \perp AB$ (*Fig.* 77), and let CE, CF, CG, be any other lines drawn from C to the line AB. These lines are the hypothenuses of the right triangles CDE, CDF, CDG, respectively, and therefore are, each of them, greater than CD (§ **82**, Corollary); therefore,—

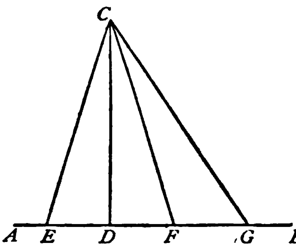


Fig. 77.

Theorem I.—*Of all lines which can be drawn from a point to a straight line the perpendicular is the shortest.*

Corollary.—Hence, *the distance from a point to a straight line is the length of the perpendicular let fall from the point to the line.*

If $DE = DF$, then $\triangle\ CDE \cong \triangle\ CDF$ (why?), therefore $CE = CF$; that is,—

Theorem II.—*Two oblique lines equally distant from the foot of the perpendicular are equal.*

Again, $DG > DF$. Likewise the angles CFD and CGD are each acute (why?). And CFG, being the supplement of CFD, must be obtuse. Therefore, in the triangle CFG we have $CFG > CGF$; hence $CG > CF$ (§ **82**, Theorem II.). That is,—

Theorem III.—*Of two oblique lines unequally distant from the foot of the perpendicular, the more remote is the greater.*

Exercises.—**1.** In each of these three theorems what is the hypothesis? What the conclusion?

2. State and prove the converse of Theorem II.

§ **84.** Let ABC and ABD (*Fig. 78*) be two isosceles triangles having a common base AB. Join CD. The triangles ACD and BCD are equal (IV. Law of Equality). Let the triangle ACD be folded over CD until it coincides with the triangle CBD; then A will fall on B, and all the lines and angles which fall upon each other will be equal. Hence we have, (*i.*) $a = b$ and $c = d$; (*ii.*) $AE = BE$; (*iii.*) $m = n$, and therefore $CD \perp AB$.

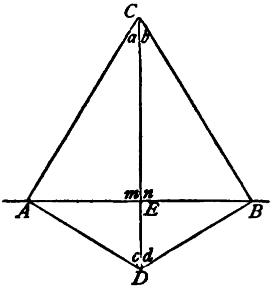


Fig. 78.

Theorem. — *If two isosceles triangles have a common base, the line drawn through their vertices* (i.) *bisects the angles at the vertices;* (ii.) *bisects the base;* (iii.) *is perpendicular to the base.*

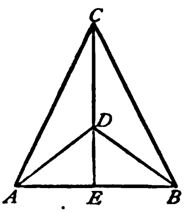


Fig. 79.

Exercises. — 1. Prove this theorem, when (as in *Fig. 79*) the vertices of the two isosceles triangles are both on the same side of the base.

2. If any number of isosceles triangles have a common base, the line passing through the vertices of two of the triangles will pass through the vertices of all the triangles.

3. If (*Fig. 79*) $CAE = 64°$ and $DAE = 36°$, find all the other angles in the figure.

§ **85. Problem.** — *To bisect a given angle* BAC (*Fig. 80*).

Analysis. — The theorem of § **84** suggests a mode of solving this problem. If we construct an isosceles triangle having the given angle BAC for the angle at the vertex, and then upon its base construct any other isosceles triangle, the line which joins the vertices of the two triangles must, by § **84** (i.), be the bisector required.

Construction.—With A as centre describe an arc cutting the sides of the given angle in M and N. With M and N as centres, and equal radii, describe arcs intersecting at a point D. Join $A D$: it is the bisector required.

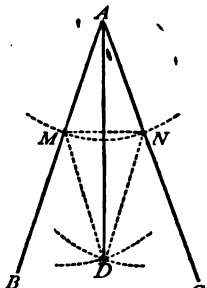


Fig. 80.

Exercises.—**1.** Prove, directly, that $\triangle AMD \cong \triangle AND$ (*Fig. 80*), and hence that $BAD = CAD$.

2. Make an acute angle, and then bisect it.

3. Make an obtuse angle, and then bisect it.

4. Draw a triangle, and then bisect the three angles. In how many points do the bisectors meet one another?

5. Divide an angle into four, and also into eight, equal parts.

6. Make, with ruler and compasses, an angle of 30°; an angle of 45°.

FORM. END

§ **86.** **Problem.**—*To bisect a given straight line $A B$ (Fig. 81).*

Analysis.—If we construct upon $A B$ as a base any two isosceles triangles, the line joining their vertices must, by § **84** (ii.), bisect $A B$.

Construction.—With A and B as centres and equal radii, describe arcs intersecting above $A B$ in C, and below $A B$ in D. Join $C D$: it bisects $A B$ in E.

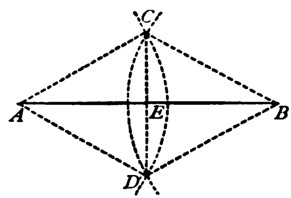


Fig. 81.

Exercises.—**1.** Bisect a line, first freehand, then with ruler and compasses.

2. Bisect the three sides of a triangle, and then join the points of bisection to the vertices of the opposite angles. In how many points do the joining lines (called the *medians* of the triangle) intersect?

3. Divide a line into four, and also into eight, equal parts.

4. Can you, by means of this problem, divide a line into six equal parts?

5. Bisect a line more than twice as long as the greatest opening of the compasses?

§ **87.** From § **84** it appears that every point in the perpendicular erected at the middle of a line is equally distant from the ends of the line. Moreover, every point *not* in this perpendicular is unequally distant from the ends of the line.

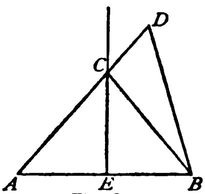


Fig. 82.

For let D (*Fig. 82*) be such a point, CE being the perpendicular. Since $AD = AC + CD = BC + CD$, and since also $BC + CD > BD$ (why?), therefore $AD > BD$.

Hence, —

Theorem. — *Every point equidistant from the ends of a straight line is in the perpendicular which bisects the line.*

§ **88.** **Problem.** — *To erect a perpendicular at a given point C (Fig. 83) of a given line AB.*

Analysis. — If we find in AB two points, M and N, equally distant from C, and then find any other point D equidistant from M and N, it is evident from § **87** that the line passing through C and D will be the perpendicular required.

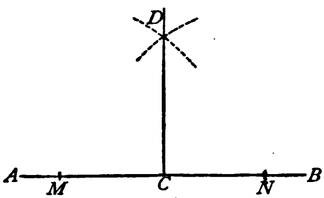


Fig. 83.

Construction. — With the centre C describe an arc cutting AB in M and N. With M and N as centres and equal radii, describe arcs intersecting at a point D. Join CD.

Note. — In practice this construction is usually effected by means of the ruler and the square (see § **49**).

Exercises. — **1.** In the above construction what is the least value which the equal lines MD and ND can have?

2. Erect a perpendicular at the end of a line without prolonging the line.

Construction. — Let AC (*Fig. 84*) be the line. Construct on AC as base an equilateral triangle ABC (how is this done?), and prolong AB to D, making $BD = AB$. CD is the perpendicular required.

Proof. — Since $\triangle$ ABC is equilateral, $ABC = 60°$; therefore, in $\triangle$ BCD, $BCD + BDC = 60°$ (§ **66**). Now $\triangle$ BCD is isosceles (why?);

hence $BCD = BDC = 30^\circ$. Therefore $ACD = ACB + BCD = 60^\circ + 30^\circ = 90^\circ$.

3. Construct an equilateral triangle having the altitude 40^{cm}.

Hints. — Draw a line 40^{cm} long, erect at one end a perpendicular, at the other end make two angles of 30° each.

4. Construct an isosceles triangle having given (*a*) the base and the altitude; (*b*) a side and the altitude.

5. Bisect the three sides of a triangle, and then erect perpendiculars at the points of bisection. In how many points do they intersect?

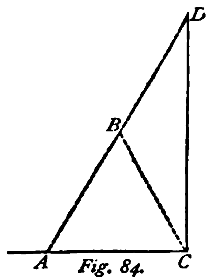


Fig. 84.

§ **89. Problem.** — *To let fall a perpendicular from a given point C (Fig. 85) to a given line A B.*

The analysis and the construction are the same as in the last problem, the only difference being that in this case the point C is not on the line AB. Give the analysis and the construction in full.

Note. — This construction is also usually effected in practice by means of the ruler and the square (§ **49**).

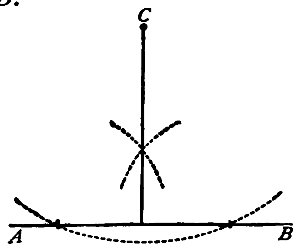


Fig. 85.

Exercises. — 1. Choose a point on each side of a line, and let fall perpendiculars from each point to the line.

2. Draw a triangle, and from the vertex of each angle let fall a perpendicular to the opposite side. In how many points do the perpendiculars intersect?

3. Through a given point draw a parallel to a given line by constructing two perpendiculars.

§ **90. Theorem.** — *Two parallel lines are everywhere equally distant from each other.*

Proof. — From any two points A and B (*Fig. 86*) of one of the

lines let fall perpendiculars meeting the other line in C and D; then $AC \parallel BD$ (§ **57**, Corollary). Join BC. The triangles ABC and BCD are equal (§ **56** and I. Law of Equality). Therefore $AC = BD$; that is, *any* two points of one line are equally distant from the other line; hence the lines are everywhere equally distant from each other.

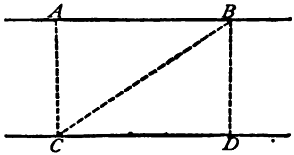


Fig. 86.

§ **91.** Let the problem be proposed: *to find a point at a given distance from a given point.*

It is evident that any point in the circumference of a circle, described with the given point as centre and the given distance as radius, will satisfy the required condition, and will, therefore, be a solution of the problem. It appears, then, that there are an indefinite number of solutions to the problem; hence the problem is termed *indeterminate.*

The required point is limited in position to a certain *line;* this line is called the *locus* of the point.

Definition. — *The line (or lines) in which a point must be, in order to satisfy a given condition, is called the* LOCUS[1] *of the point.*

Again, let it be proposed: *to find a point equidistant from two given points.*

It follows from § **87** that the required point may be anywhere in the perpendicular which bisects the straight line joining the given points. The problem, therefore, is indeterminate, and the locus of the point required is the above-mentioned perpendicular.

Generally speaking, when a point has to satisfy only *one* geometrical condition, the problem is indeterminate, and the solution consists in finding the locus of the required point.

---

[1] Plural *loci.*

Exercises.—1. In the cases above considered, the points were supposed to be confined to one plane: what would the loci be if this restriction were removed?

Ans. In the first case, the surface of a sphere with the given point as centre and the given distance as radius; in the second case, the plane perpendicular to the line which joins the given points, and bisecting it.

2. Find the locus of a point equidistant from a given straight line.

3. Find the locus of a point equidistant from a given circumference.

4. Find the locus of a point equidistant from a given plane surface.

5. Find the locus of a point equidistant from two given parallel lines.

6. Find the locus of a point equidistant from two given intersecting lines.

Hints.—The locus consists of the bisectors of the angles made by the lines. To prove this, let P be *any* point on either of the bisectors, PM and PN perpendiculars from P to the two lines, O the point where the lines meet; then show that $\triangle POM \cong \triangle PON$, and therefore that $PM = PN$. Why is it that, although there are *four* angles to be bisected, the locus will consist of only *two* straight lines?

§ **92.** When a point has to be found which satisfies *two* conditions, the problem (if possible at all) generally has *one* or *two* solutions; in other words, the problem is *determinate.*

Such problems are solved by constructing the loci of points which satisfy each condition separately; then the point, or points, where the loci intersect, being common to both loci, will satisfy both conditions, and will be the solution of the problem.

For example: *find a point which shall be at given distances from two given points.*

The required point must be in a circumference having the first given point for centre and the first given distance for radius; and it must also be in a circumference having the second given point for centre and the second given distance for radius; hence it must be at the intersection of these two circumferences.[1]

Since, in general, these circumferences intersect in two points, there will be in general two solutions of the problem. When will there be only one solution? When will there be no solution at all?

In the following exercises, after having given the solution, state under what conditions (if any) the problem is impossible.

---

[1] This reasoning has been already employed in § **75.**

Exercises. — **1.** Find a point in a given straight line at a given distance from a given straight line.

2. Find a point in a given straight line at equal distances from two other straight lines.

3. Construct an isosceles triangle having a given base, and each of its sides equal to three times the base.

4. Find a point at a given distance from a given point, and at the same distance from a given straight line.

5. Construct a triangle having given the base, the sum of the other sides, and one of the angles at the base.

6. Construct a triangle having given the base, the difference of the other sides, and one of the angles at the base.

7. Find a point which shall be equidistant from three given points.

8. Find a point which shall be equidistant from three given intersecting straight lines (see § **91**, Exercise **6**). Show that there are four solutions, and construct the four points. How is the solution modified if two of the given lines are parallel?

§ **93.** The complete solution of a problem consists of three parts: —

(*i.*) The ANALYSIS, in which we explain how a correct construction can be based upon known geometrical truths.

(*ii.*) The CONSTRUCTION, in which, guided by the analysis, we make the required lines, angles, etc., with the help of the ruler and the compasses.

(*iii.*) The DISCUSSION, in which we consider (*a*) whether there may be more than one solution; (*b*) whether the data may have such values that no solution is possible; (*c*) whether the solution has any other peculiarities for particular values of the data.

The analysis of a problem that can be solved by the method of loci has already been given (§ **92**); in all other cases a theorem (or theorems) must first be found on which a correct construction can be based. In this search nothing but experience and ingenuity will ensure success; but there is one general rule which will be found very useful; namely: —

Rule.—*Suppose the solution effected, and draw a suitable figure; then trace the relations among the parts of the figure until some relation is discovered which will give a clue to the right construction.*

We will give an example to illustrate the application of this rule.

Problem.—*Through a given point to draw a line which shall cut off equal lengths from the sides of a given angle.*

Analysis.—Let P (*Fig.* 87) be the given point, $B\,A\,C$ the given angle. Suppose that $O\,P\,Q$ were the line required; then we know that $A\,O\,Q$ must be an isosceles triangle (why?), and hence, that if we bisected the angle A, $O\,P\,Q$ would be perpendicular to the bisector (§ **80**, Exercise **1**). This suggests to us the correct construction.

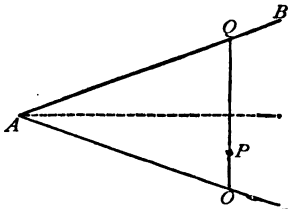


Fig. 87.

Construction.—Bisect the angle A, let fall from P a perpendicular to the bisector, and produce it until it meets the sides of the given angle in O and Q: $O\,P\,Q$ is the line required.

Discussion.—Construct the figure for the case where P is not between the sides of the given angle. For what position of P is the solution impossible?

Exercises.—**1.** Draw a line which shall pass through a given point and make equal angles with two given intersecting lines.

2. In a given straight line find a point which shall be equidistant from two given points.

3. In one side of a triangle find a point which shall be equidistant from the other two sides (see § **91**, Exercise **6**).

4. In one side of an angle a point is given; find in the same side another point which shall be equidistant from the first point, and from the other side of the angle.

5. From a given point without a given straight line draw a line making a given angle with the given line.

6. Trisect a right angle (that is, divide it into three equal parts).

VI.—Applications.

§ 94. If I wish to find the distance from A to B (*Fig. 88*) it is plain that the intervening pond will prevent me from using a chain or other direct means of measurement.

In numerous instances direct measurement would be very troublesome or quite impossible. On the ground obstacles, such as houses, water, swamps, are often met with; if we want to measure the distance from the earth to the moon, only one end of the line to be measured is accessible; while in the case of the distance between the sun and a planet, the line is wholly inaccessible.

Here Geometry comes to our aid by giving us methods of *indirect* measurement; in which, for example, by measuring one line we are able to learn the length of another.

How this can be done by the help of the laws of equal triangles, we are now prepared to understand.

All cases of the indirect measurement of a line may be reduced to four,—

(*i.*) Both ends of the line are accessible;
(*ii.*) Only one end is accessible;
(*iii.*) Both ends are inaccessible;
(*iv.*) The line is wholly inaccessible.

§ 95. Problem.—*To measure a line the ends of which only are accessible.*

Method I.—Let $A\,B$ (*Fig. 88*) be the line to be measured. Choose a point C from which A and B are both visible. Measure $A\,C$, $B\,C$, and the angle $A\,C\,B$.

Prolong $A\,C$ to D, making $C\,D = C\,A$, and prolong $B\,C$ to E, making $C\,E = C\,B$. $\triangle\,D\,E\,C \cong \triangle\,A\,B\,C$ (why?). Therefore $D\,E = A\,B$, and its length (which can be measured directly) is the distance required.

Method II. — Measure $A\,C$, $B\,C$, and $A\,C\,B$, as before. Then construct on paper an angle $a\,c\,b = A\,C\,B$; choose a suitable scale of reduction, and lay off on the paper the reduced lengths $a\,c$ and $b\,c$ of the lines $A\,C$ and $B\,C$. Join $a\,b$: its length gives, *to the reduced scale*, the distance $A\,B$ required.

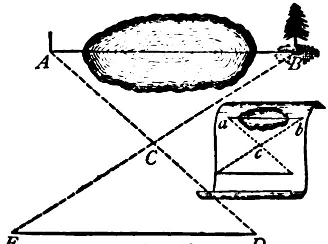


Fig. 88.

For example: if $A\,C = 800^{m}$, $B\,C = 1000^{m}$, and the scale of reduction is 1 : 5000, then $a\,c = 16^{cm}$, and $b\,c = 20^{cm}$. If now we find that $a\,b = 24^{cm}$, then $A\,B = 24^{cm} \times 5000 = 1200$ metres.

Remark. — The first method is preferable, because any error made in measuring $a\,b$ is multiplied 5000 times in the result.

How can the length of the pond (*Fig. 88*) be found?

§ **96. Problem.** — *To measure a line one end only of which is accessible.*

Method I. — Let $A\,B$ (*Fig. 89*) be the line to be measured, B the end which is accessible.

At a point C, in $A\,B$ prolonged, place a signal (a pole or flag). Then place a signal at a convenient point D, and measure the distances $B\,D$ and $C\,D$. Prolong $C\,D$ to E, making $DE = D\,C$, and prolong $B\,D$ to F, making $D\,F = D\,B$. Place signals at E and F. Then proceed in the direction $E\,F$ until a

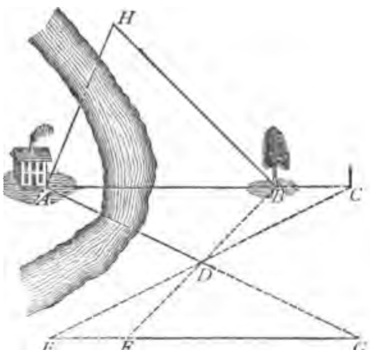


Fig. 89.

point G is reached which falls in the line AD. Measure FG: its length is equal to the distance AB required.

Proof: $\triangle DEF \cong DBC$ (II. Law of Equality). Therefore angle $DEF = DCB$; hence $EG \parallel AC$ (§ **57**, Remark). In the triangles DAB and DFG, $DB = DF$, $ADB = FDG$ (why?), and $ABD = DFG$ (why?); therefore $\triangle DAB \cong \triangle DFG$ (I. Law of Equality), and $FG = AB$.

Method II. — (By an isosceles triangle.) Choose a convenient direction BH for running a straight line from B, measure the angle ABH, and find in BH a point H at which the angle AHB shall be equal to $\frac{1}{2}(180 - ABH)$. Then $BH = AB$ (why?). What will be the value of AHB if $ABH = 90°$?

Remark. — In both methods three things have to be measured directly: in Method I., three lines; in Method II., one line and two angles. But, if the means of measuring an angle are at hand, Method II. is preferable, because it is easier to measure an angle than a line.

§ **97. Problem.** — *To measure a line when both ends of it are inaccessible.*

Let AB (*Fig. 90*) be the line, and let us suppose that a part of the line between A and B is accessible, as shown in the Figure.

Method I. — At a point C in AB erect a perpendicular CD (§ **88**), and take $DE = CD$. At E also erect a perpendicular FG; then FG will be parallel to AB (why?)

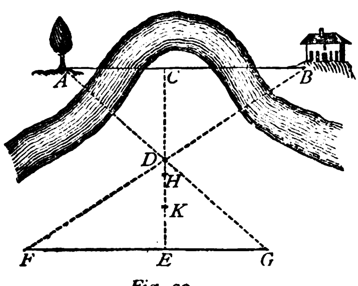


Fig. 90.

Find in FG the point F which falls in the line BD, and the point G which falls in the line AD. Now $\triangle ACD \cong \triangle EDG$ (why?), and $\triangle BCD \cong \triangle EDF$ (why?).

Therefore $FE = BC$, and $EG = AC$; and hence $FE + EG = BC + AC = AB$, or the line to be measured.

Method II. — (By isosceles triangles.) In CE find the points H and K from which the directions of A and B respectively make with CE the angle 45°. Then $CH + CK = AB$. Explain fully why.

§ **98. Problem.** — *To measure a line which is wholly inaccessible.*

Method I. — Choose a convenient point C from which A and B are both visible; measure AC and BC as in § **96**; then measure AB as in § **95**.

Method II. — Choose C as before; also a point F from which A is visible, and a point G from which B is visible. Find E, the intersection of AF and BC, and D, the intersection of BG and AC. Measure all the angles at C, and also the lines CF, CE, CG, CD. Then represent on paper the measured angles and lines (the latter to a reduced scale).

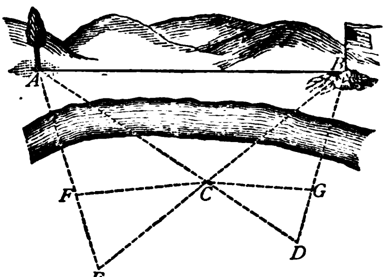


Fig. 91.

By prolonging EF and DC till they meet, we find the point on the paper which corresponds to A on the ground; and by prolonging in like manner DG and EC, we find the point which corresponds to B. The length of AB on the paper gives *to the scale employed* the required distance.

REVIEW OF CHAPTER IV.

QUESTIONS.

1. Define a *plane figure;* a *polygon;* its *perimeter.*
2. Define a *triangle*, and explain how it is named.
3. What *six parts* has every triangle?
4. Distinguish between *exterior* and *interior* angles.
5. The sum of any two sides of a triangle is greater than the third side.
6. Define *equilateral, isosceles*, and *scalene* triangles.
7. Define the terms *base, vertex*, and *altitude.* When will the altitude lie outside the triangle?
8. The sum of the angles of a triangle is equal to 180°. How can this truth be illustrated?
9. Why can a triangle have only one right or one obtuse angle?
10. Define *acute, right*, and *obtuse* triangles.
11. If two angles of a triangle are known how can the third be found?
12. In a right triangle what is the sum of the acute angles?
13. What is the *complement* of an angle?
14. An exterior angle of a triangle = the sum of the opposite interior angles.
15. Define *similar, equivalent*, and *equal* magnitudes, and give examples of each kind.
16. In two triangles what are *corresponding* sides (or angles)?
17. What is the least number of parts which determine the size and shape of a triangle? Of these how many at least must be *sides?*
18. In what cases do three parts fail to determine the triangle?
19. When are two triangles *mutually equiangular?*
20. Construct a triangle, having given, —
    (*i.*) a side and two angles;
    (*ii.*) two sides and the included angle;
    (*iii.*) two sides and the angle opposite to the greater side;
    (*iv.*) the three sides.
21. State the four laws of equal triangles.
22. Make a triangle equal to a given triangle.
23. Make an angle equal to a given angle.
24. Draw a line parallel to a given line.
25. What is a *problem?* What are *auxiliary* lines, and how distinguished from others in the construction?

26. In an isosceles triangle the angles opposite to the equal sides are equal. Corollary.
27. If two angles of a triangle are equal the triangle is isosceles. Corollary.
28. What is the *converse* of a theorem? Give an example.
29. In a triangle the greater of two angles is opposite to the greater side, and conversely. Corollary.
30. The perpendicular is the shortest line from a point to a straight line. Corollary.
31. Oblique lines equally distant from the foot of the perpendicular are equal.
32. Of two oblique lines unequally distant from the foot of the perpendicular, the more remote is the greater.
33. If two isosceles triangles have a common base the line joining their vertices (*i.*) bisects the angle at the vertices, (*ii.*) bisects the base, (*iii.*) is perpendicular to the base.
34. Bisect a given angle.
35. Bisect a given straight line.
36. Every point equidistant from the ends of a straight line is in the perpendicular bisecting the line.
37. Erect a perpendicular at a given point of a given line.
38. Let fall a perpendicular from a given point to a given line.
39. Two parallel lines are everywhere equally distant from each other.
40. Give an example of an *indeterminate* problem.
41. Define the *locus* of a point, and illustrate the definition.
42. Give an example of a *determinate* problem.
43. How are problems solved by the method of loci?
44. What are the three parts of the complete solution of a problem?
45. What general rule is often useful in solving problems?
46. What four cases may occur in the indirect measurement of a line?
47. Measure a line the ends only of which are accessible.
48. Measure a line one end only of which is accessible.
49. Measure a line neither end of which is accessible.
50. Measure a line which is wholly inaccessible.

EXERCISES.

1. In an isosceles triangle the exterior angles at the base are equal.
2. In an isosceles triangle the bisector of an exterior angle at the vertex is parallel to the base.

3. In an isosceles triangle the bisectors of the base angles (prolonged until they meet the sides) are equal.

4. In what cases are we able, knowing one angle of a triangle, to find the other two angles?

5. If in a right triangle the acute angles are 30° and 60°, the hypotenuse is equal to twice the smaller leg.

6. State and prove the converse of the preceding theorem.

7. Divide an isosceles triangle into two equal right triangles.

8. Make an angle equal to three times a given angle.

9. Make an angle of 15°; also an angle of 150°.

10. In what two ways can an angle of $22\frac{1}{2}^\circ$ be constructed?

11. Name various angles between 0° and 180° which can be constructed by means of the theorems and problems already given.

12. A straight railway passes within two miles of a town. A place is described as four miles from the town and one mile from the railway. How many places satisfy the conditions?

13. Place a line of given length between the sides of an angle so as to be parallel to a given line.

14. Through a given point between two lines not parallel draw a line which shall be bisected at that point.

15. Three lines meet in a point; draw a line such that the parts of it intercepted by the three lines shall be equal.

16. What axiom is implied in the last line of the proof of the theorem in § **87**?

17. The perpendiculars erected at the middle points of the sides of a triangle meet in one point.

Hints.— Erect two of the perpendiculars; show (by II. Law of Equality) that their intersection is equidistant from the three corners of the triangle, then make use of § **80**, Exercise 2.

18. The bisectors of the three angles of a triangle meet in one point.

Hints.— Draw two bisectors; show by § **91**, Exercise 6, that their intersection is equidistant from the sides of the triangle, and (by III. Law of Equality), that the line joining the intersection with the third corner bisects the angle at that corner.

19. Trisect a given straight line.

20. Illustrate by an example the application of the rule in § **93**.

21. Choose two stations on the ground, and find their distance by means of (*i.*), § **95**; (*ii.*), § **96**; (*iii.*), § **97**; (*iv.*), § **98**.

22. Can you find the height of an object (a tree, a church-spire, etc.) by means of truths established in this chapter? How?

23. How many and what parts are sufficient to determine a right triangle? an isosceles triangle? an equilateral triangle? an isosceles right triangle?

CHAPTER V.

QUADRILATERALS.

CONTENTS. — I. Sides and Angles of a Quadrilateral (§§ 99, 100). II. Different Kinds of Quadrilaterals (§§ 101–105). III. Construction of Quadrilaterals (§§ 106–111). IV. Subdivision of a Line (§§ 112–114).

I. — Sides and Angles of a Quadrilateral.

§ **99.** A plane figure bounded by four straight lines is called a QUADRILATERAL (*ABCD, Fig. 92*).

A straight line AC, which joins two opposite corners of a quadrilateral, is called a DIAGONAL.

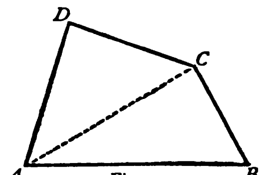


Fig. 92.

Exercises. — 1. How many sides and how many angles has a quadrilateral?

2. How is a quadrilateral named or denoted?

3. What is the *perimeter* of a quadrilateral (§ 59)?

4. Into how many triangles is a quadrilateral divided by a diagonal?

5. How many diagonals can be drawn in a quadrilateral?

§ **100.** If we divide a quadrilateral $ABCD$ (*Fig. 92*) into two triangles by drawing a diagonal AC, it is evident that the sum of the four angles of the quadrilateral is the same as the sum of the six angles of the two triangles. Now the sum of the angles of the two triangles is 360° (why?); hence, —

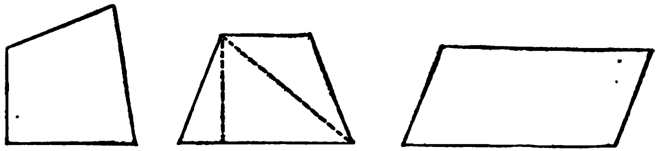
Theorem. — *The sum of the angles of a quadrilateral is equal to four right angles, or 360°.*

Exercises. — 1. If the angles of a quadrilateral are all equal, what is the value of each?

2. Three angles of a quadrilateral are 70°, 40°, and 120°; find the fourth.

II.—Different Kinds of Quadrilaterals.

§ **101.** With respect to the position of the sides there are three kinds of quadrilaterals.



I. II. III.

Fig. 93.

A quadrilateral which has no sides parallel is called a TRAPEZIUM (*Fig. 93*, I.).

A quadrilateral which has only two sides parallel is called a TRAPEZOID (*Fig 93*, II.).

A quadrilateral which has its opposite sides parallel is called a PARALLELOGRAM (*Fig. 93*, III.).

Exercises.—1. Draw (free-hand) a trapezium, a trapezoid, and a parallelogram.

2. Draw a trapezoid having two right angles.

§ **102.** Let $ABCD$ (*Fig. 94*) be a parallelogram. Draw the diagonal BD. The alternate angles m and n are equal, and likewise the alternate angles p and q (§ **56**). Therefore $\triangle ABD \cong \triangle CBD$ (I. Law of Equality), and $AB = CD$, and $AD = BC$. Also the angle A equals the angle C; and, since $m = n$ and $p = q$, therefore $m + n = p + q$; that is, the angles at B and D are equal. Hence,—

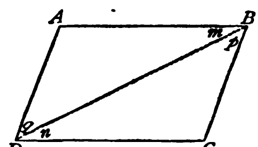


Fig. 94.

Theorem I.—*Every parallelogram is divided by a diagonal into two equal triangles.*

Theorem II. — *The opposite sides of a parallelogram are equal.*

Theorem III. — *The opposite angles of a parallelogram are equal.*

Corollaries. — From Theorem II. it follows that, —

1. *Parallels between parallels are equal.*

2. *Perpendiculars between parallels are equal* (§ **57**, Corollary).

3. *Parallels are everywhere equally distant* (see § **90**).

Exercises. — **1.** If two adjacent sides of a parallelogram are equal, what follows as to the other sides?

2. What relation exists between two adjacent angles of a parallelogram? Why?

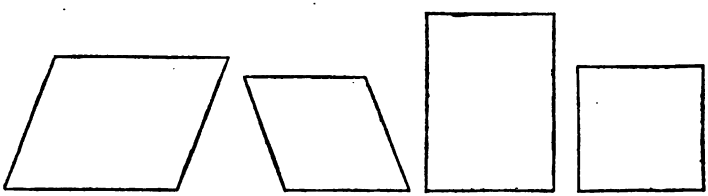
3. If one angle of a parallelogram is a right angle, what must the others be? Illustrate with a figure.

4. If one angle of a parallelogram is acute, what can be inferred of the other angles? Illustrate with a figure.

5. One angle of a parallelogram is (*i.*) 45°, (*ii.*) $54^\circ\ 18'$, (*iii.*) $121^\circ\ 16'\ 44''$; find the other angles.

6. Prove the converse of Theorem II.

§ **103.** The sides of a parallelogram may be either equal or unequal, and the angles may be either right or oblique; so there are in all four kinds of parallelograms: the *oblique unequal-sided*



I. II. III. IV.

Fig. 95.

parallelogram or Rhomboid (*Fig. 95*, I.), the *oblique equal-sided* parallelogram or Rhombus (*Fig. 95*, II.), the *right unequal-sided*

parallelogram or RECTANGLE (*Fig. 95*, III.), and the *right equal-sided* parallelogram or Square (*Fig. 95*, IV.).

Exercises. — **1.** Give examples of parallelograms (surface of a table, roof of a house, etc.), and state the kind in each case.

2. Draw (free-hand) a parallelogram of each kind.

3. One angle of a rhombus is 60°; find the other angles.

4. One side of a rhombus is 24^m; find its perimeter.

5. The sides of a rectangular park are 400^m and 240^m. How many trees must be set out in its perimeter if they are to stand 10^m apart?

§ **104.** In the parallelogram $ABCD$ (*Fig. 96*), draw the diagonals AD and BC; how many triangles are thus formed? $\triangle AOB \cong COD$ (why?); therefore the sides opposite to the equal angles are equal; that is, $AO = DO$ and $BO = CO$; hence, —

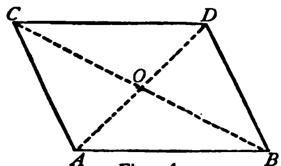


Fig. 96.

Theorem. — *The two diagonals of a parallelogram mutually bisect each other.*

Exercises. — **1.** Find the angles which the diagonals of a square make with the sides of the square.

2. In a square the diagonals are equal and perpendicular to each other.

3. In a rectangle the diagonals are equal and inclined to each other.

Hint. — Draw the diagonals, and make use of § **73**.

4. In a rhombus the diagonals are perpendicular to each other. Are they equal or unequal?

5. In a rhomboid the diagonals are inclined to each other. Are they equal or unequal?

6. What kind of parallelograms are divided by the diagonals into isosceles triangles? Into right triangles?

§ **105.** Either side of a parallelogram may be regarded as the *base;* then a perpendicular let fall to the base from any point in the opposite side is the *altitude* of the parallelogram.

In the parallelogram $ABCD$ (*Fig.* 97) AB is taken as the base, and either DE or FB or CG is equal to the altitude. Why are DE, FB, and CD all equal?

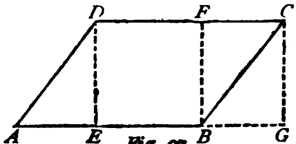


Fig. 97.

In a rectangle whatever side be taken as base the adjacent side is the altitude.

In a square the base and altitude are equal.

In a trapezoid one of the parallel sides is taken as the base.

In a trapezium the terms base and altitude are not used.

III.—Construction of Quadrilaterals.

§ **106.** **Problem.**—*To construct a square, having given one side, a* (*Fig.* 98).

Construction.—At A make a right angle; take $AB = AD = a$; then from B and D as centres, with a radius equal to a, describe arcs intersecting at a point C. $ABCD$ is the square required.

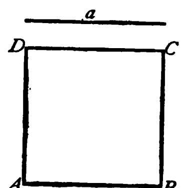


Fig. 98.

Proof.[1] The four sides are each equal to a by construction. Join BD, and make use of § **102**, Theorem I.

Construct another square with the same side a; then prove (by division into triangles, and IV. Law of Equality) that the two squares are equal.

Hence, *a square is completely determined if one side is known;* and,—

Theorem.—*Two squares are equal if they have a side equal.*

Exercises.—**1.** Construct a square whose side shall be 1^{dm}.

2. Construct a square whose perimeter shall be 1^{m}.

3. Draw a rectangle; then construct a square with the same perimeter as that of the rectangle.

---

[1] If the construction of a problem is given without a previous analysis, a proof of the construction (unless it is quite obvious without it) ought to follow.

4. Construct a square, having given its diagonal.
5. Construct a square, and upon each side an equilateral triangle.
6. Construct an equilateral triangle, and upon each of its sides a square.

§ **107.** **Problem.** — *To construct a rectangle, having given two adjacent sides, a and b (Fig. 99).*

Construction and proof similar to those of the last section.

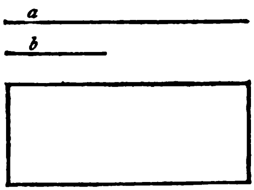


Fig. 99.

Construct another rectangle with the same sides, and prove that the two rectangles are equal.

Hence, *a rectangle is determined if we know two adjacent sides;* also, —

Theorem. — *Two rectangles are equal if they have two adjacent sides equal each to each.*

Exercises. — **1.** Construct a rectangle with the sides 40^{cm} and 60^{cm}. (If on paper, use a reduced scale.)

2. Make a plan of a rectangular field whose sides measure 640^{m} and 360^{m}.

3. How many rectangles can be constructed upon a given line as the diagonal?

§ **108.** **Problem.** — *To construct a parallelogram, having given two adjacent sides and the included angle.*

Construction. — Let a and b (*Fig. 100*) be the given sides, 70° the included angle. Make an angle $A = 70°$, then proceed as in the last section. $ABCD$ is the required parallelogram.

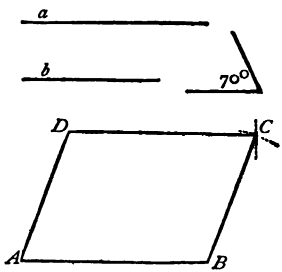


Fig. 100.

Proof similar to that in § **106**.

Construct a second parallelogram with the same given parts, and prove that it is equal to the first.

Therefore, *two sides and the included angle completely determine a parallelogram;* and, further, —

Theorem. — *Two parallelograms are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Exercises. — **1.** Construct a parallelogram, the two sides being 44^{cm} and 66^{cm}, and the included angle 110°.

2. Construct a rectangle, having given —

(*a*) A side and a diagonal;
(*b*) A side and the opposite angle of the diagonals;
(*c*) A diagonal and the angle between the diagonals.

3. Construct a rhombus, having given —

(*a*) A side and an angle;
(*b*) A side and a diagonal;
(*c*) The diagonals;
(*d*) An angle and the diagonal through its vertex.

4. Construct a rhombus whose angles shall have the ratio $1:3$.

§ **109. Problem.** — *To construct a trapezoid, having given three sides (two being the parallel sides) and one angle.*

Construction. — Let a, b, c (*Fig. 101*) be the given sides, 68° the given angle.

Make an angle $A = 68^\circ$; take $AB = a$, $AD = b$; through D draw a line parallel to AB, and on it take $DC = c$. Join BC. $ABCD$ is the trapezoid required.

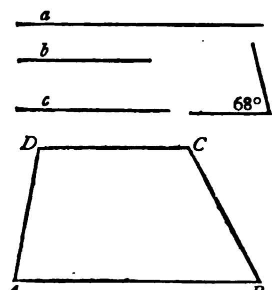


Fig. 101.

Exercises. — **1.** How many and what parts determine a trapezoid?

2. Construct a trapezoid with the sides 60^{cm}, 70^{cm}, 80^{cm}, and the angle between the first two sides equal to 75°.

3. Construct a trapezoid, having given —

(*a*) The two parallel sides and the altitude;
(*b*) The two sides not parallel and the altitude;
(*c*) The two parallel sides and one of the other sides;
(*d*) One of the parallel sides and the two sides not parallel.

§ **110. Problem.** — *To construct a trapezium, having given three sides and the two angles included by these sides.*

Construction. — If a, b, and c are the given sides, 60° and 75° the included angles, draw a line $AB = b$; through A and B draw lines making, with BB, the angles 60° and 75° respectively; lay off on these lines the lengths $AD = a$, and $BC = c$, and join the points thus found.

Draw a figure to illustrate this construction.

Exercises. — **1.** Construct a trapezium with the sides 120^{cm}, 90^{cm}, 70^{cm}, and the angles 67° and 83°.

2. Construct a trapezium, having given —

(*a*) Three angles and the two sides between the first and third angles;

(*b*) Four sides and one angle;

(*c*) Four sides and a diagonal.

§ **111. Problem.** — *To construct a quadrilateral equal to a given quadrilateral $ABCD$ (Fig. 102).*

Analysis. — If we draw the diagonal BD, we divide the given quadrilateral into two triangles. If now we construct (§ **76**) two new triangles equal respectively to these, and *similarly placed*, it is evident that they will form, taken together, a new quadrilateral equal to the given quadrilateral (Axiom II.).

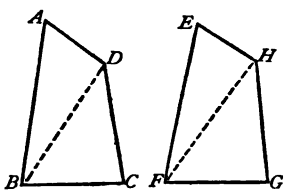


Fig. 102.

Construction. — Take $EF = AB$; from E and F as centres, with the radii AD and BD respectively, describe arcs intersecting at a point H; then from F and H as centres, with BC and DC respectively as radii, describe arcs intersecting at a point G. Lastly, join EH, HG, and GF.

Exercises. — **1.** Draw a trapezium, and then construct another equal to it.

2. When are two quadrilaterals equal to each other?

IV.—Subdivision of a Line.

§ **112.** Let a, b, c, d, e (*Fig. 103*) be parallel lines equidistant from one another, and let them be cut by any other lines as in the figure.

The perpendiculars AF, BG, CH, DK are equal (why?). $\triangle ABF \cong \triangle BCG \cong \triangle CDH \cong \triangle DEK$ (why?); therefore $AB = BC = CD = DE$.

In like manner we can prove that the parts cut from any other line, MN or PQ, are equal to each other. Hence,—

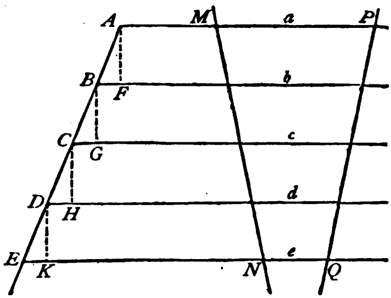


Fig. 103.

Theorem.—*Equidistant parallel lines cut equal parts from any line that intersects them.*

Exercise.—In a right triangle the middle point of the hypotenuse is equidistant from the three corners of the triangle.

Hint.—Through this middle point draw a line parallel to one of the legs.

§ **113. Problem.**—*To divide a straight line AB (Fig. 104) into any number (say five) equal parts.*

Construction.—Through A draw any straight line AX, on which lay off five equal parts of any length. Join C, the last point thus obtained, to B, and through the other points of division of AC draw lines parallel to BC. They will divide AB into five equal parts.

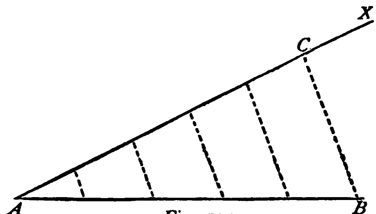


Fig. 104.

Proof. — Make use of the theorem in § **112**.

Exercise. — Divide a straight line into 3, 6, 7, 9, 10 equal parts.

§ **114.** **Problem.** — *To divide a straight line AB (Fig. 105) into two parts which shall have a given ratio (say 3 : 4).*

Construction. — Draw through A any line AC, and lay off on it $3 + 4 = 7$ equal parts. Join the last point C to B, and draw through D, the third point of division from A, a line DE parallel to BC; it intersects AB in E. Then,

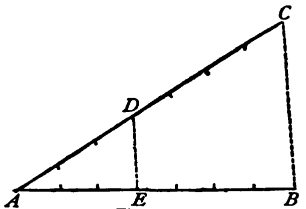


Fig. 105.

$$AE : EB = 3 : 4.$$

Proof. — Lines drawn through the other points of division of AC, and parallel to BC, would (by § **112**) divide AE into three equal parts, and EB into four equal parts.

Exercises. — **1.** Divide a line in the ratio 4 : 7.

2. Construct a rectangle, having given that the sum of two adjacent sides is 16^m, and their ratio 2 : 3.

3. Construct a rhombus in which the sum of the diagonals shall be 12^m, and one of them shall be twice the other.

4. The perimeter of a rectangular garden is 80.4^m, and the longer side is to the shorter as 5 : 7. Make a plan of the garden, and find the lengths of its sides.

5. Divide a line into three parts which shall be to each other as 2 : 3 : 5.

Hint. — Lay off on the auxiliary line $2 + 3 + 5 = 10$ equal parts.

6. Construct a triangle whose sides shall be to each other as the numbers 3, 4, 5. Is this problem determinate or not?

7. Construct an isosceles triangle whose base shall be to one of the equal sides as 3 : 4.

8. Divide a line into four parts which shall be to each other as the numbers 1, 3, 4, 6.

REVIEW OF CHAPTER V.

SYNOPSIS.

1. A *quadrilateral* is a four-sided polygon.
2. The sum of its angles equals 360°.
3. There are three classes of quadrilaterals:—
  (*i.*) The *trapezium* (no parallel sides);
  (*ii.*) The *trapezoid* (two parallel sides);
  (*iii.*) The *parallelogram* (both pairs of sides parallel).
4. There are four kinds of parallelograms:—
  (*i.*) The *rhomboid* (adjacent sides unequal, angles oblique);
  (*ii.*) The *rhombus* (sides all equal);
  (*iii.*) The *rectangle* (angles all right angles);
  (*iv.*) The *square* (sides equal, and angles right angles).
5. A diagonal divides a parallelogram into two equal triangles.
6. Hence, the opposite sides and also the opposite angles of a parallelogram are equal.
7. The two diagonals of a parallelogram mutually bisect each other, and are equal to each other in the rectangle and the square, perpendicular to each other in the rhombus and the square.
8. Two squares are equal if they have a side equal; two rectangles, if their adjacent sides are respectively equal; any two parallelograms, if they have two sides and the included angle equal each to each.
9. Equidistant parallel lines cut equal parts from any line that intersects them.
10. No. 9 enables us to divide a line into equal parts, or into parts which shall be to each other in given ratios.

EXERCISES.

1. Divide a quadrilateral into a parallelogram and a triangle.
2. Divide a parallelogram into two equal triangles.
3. Divide a parallelogram into two equal parallelograms.
4. Divide a square into four equal isosceles triangles.
5. Divide a rhombus into four equal right triangles.
6. Divide a trapezoid in which the non-parallel sides are equal into a parallelogram and an isosceles triangle.

7. Divide a right triangle into two isosceles triangles.

8. The adjacent angles of a parallelogram are supplementary.

9. In a rhomboid one angle is 120°; find the other angles.

10. If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram. (Draw a diagonal. Use §§ **56, 57, 73.**)

11. The middle points of the four sides of a parallelogram are the corners of a new parallelogram. If the first parallelogram is a rectangle the second is a rhombus; if the first is a rhombus the second is a rectangle; if the first is a square the second is a square.

12. The bisectors of the angles of a parallelogram enclose a rectangle. (Exercise 8 and § **64.**)

13. Prove the converse of § **102,** Theorem III.

14. The three perpendiculars let fall from the corners of a triangle meet in one point.

Hints.—Through the corners of the triangle draw lines parallel to the opposite sides, and prolong them till they intersect, forming a new triangle. Then show (by means of § **102,** Theorem II., Axiom I., and § **56,** Corollary) that the three altitudes of the given triangle are perpendicular to the middle points of the new triangle, and make use of Exercise 17, page 110.

15. Construct a rectangle, having given one side and the perimeter.

16. Construct a rhombus which shall have its obtuse angles twice as large as those which are acute.

17. Construct a rhomboid, having given—

(*a*) Two unequal sides and the included angle;

(*b*) Two unequal sides and a diagonal;

(*c*) One side and the two diagonals.

18. Place a line of given length between two given parallel lines so that it shall pass through a given point. (Three different positions of the point.)

19. Place a square within a given square so that one corner of it shall be at a given point in a side of the given square.

20. In a square construct an equilateral triangle having one corner common with a corner of the square, and the others in the sides of the square.

21. How many and what parts (sides or angles) are required to determine a square? a rectangle? a rhombus? a parallelogram? a trapezoid? a trapezium?

22. Divide a line into $3\frac{1}{2}$ parts.

23. Construct a rectangle whose perimeter shall be 1^m, and the ratio of the base and altitude $3:7$. What are the lengths of its sides?

CHAPTER VI.

POLYGONS.

CONTENTS.—I. Sides and Angles of a Polygon (§§ 115–117). II. Regular Polygons (§§ 118–122). III. Symmetrical Figures (§ 123).

I.—Sides and Angles of a Polygon.

§ **115.** A POLYGON is a plane figure bounded by straight lines. What are its *sides* and its *perimeter?* (§ **59**).

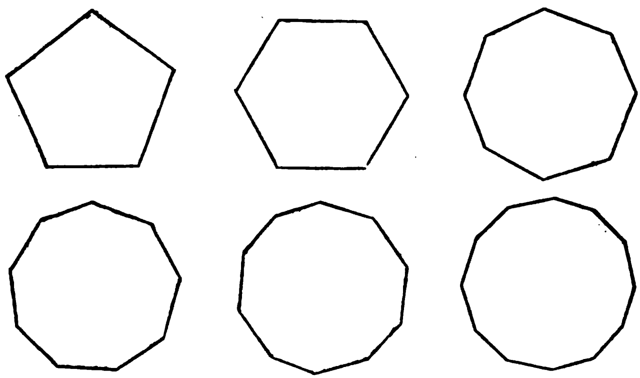


Fig. 106.

As regards the number of sides, polygons begin with three (§ **60**), and go on increasing without limit. As the number of sides increases, the polygon approaches a circle in shape (see *Fig. 106*); in other words, the circle is the limit which the polygon approaches (but never reaches) when the number of its sides is increased more and more.

The triangle and the quadrilateral are polygons; but, as they have been already studied, we shall now apply the term chiefly to those figures which have more than four sides. A polygon of five sides is called a PENTAGON, of six sides a HEXAGON, of seven sides a HEPTAGON, of eight sides an OCTAGON, of nine sides a NONAGON, of ten sides a DECAGON, of twelve sides a DODECAGON, of twenty sides an ICOSAGON.

Which of these polygons are represented in *Fig. 106?*

Every polygon has as many angles as sides; each side has two adjacent angles; each angle is formed by two intersecting sides.

A line joining two corners not in the same side is called a DIAGONAL.

By drawing as many diagonals as possible from one corner of a polygon we divide the polygon into triangles; and Exercise 1, below, leads to the following general law: —

The number of diagonals which can be drawn from one corner of a polygon is THREE *less than the number of sides; and the number of triangles into which the polygon is divided is* TWO *less than the number of sides.*

Exercises. — 1. Draw polygons of 3, 4, 5, 6, 7, 8 sides, and from a corner of each all the diagonals possible; then prepare three vertical columns, in the first writing the number of sides of the several polygons, in the second the number of diagonals, in the third the number of triangles. What general conclusions follow from the results?

2. How many diagonals can be drawn from one corner of a polygon of 60 sides, and how many triangles will be formed?

3. Divide an octagon into three quadrilaterals and a triangle.

4. If you cut off a triangle from a hexagon with a diagonal, what kind of polygon is left?

5. How many diagonals *in all* can be drawn in a quadrilateral? a pentagon? a hexagon? a decagon?

§ **116.** The angles of a polygon may be acute, right, obtuse, or even convex. In the last case, they are usually termed *reëntrant*

angles. In *Fig. 107*, *A* is a polygon having one reëntrant angle, and *B* a polygon having two reëntrant angles.

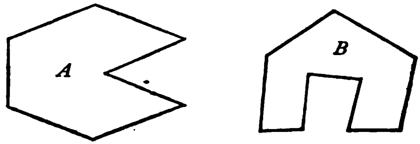


Fig. 107.

But whatever be the values of the angles, their sum in any polygon always bears a simple relation to the number of sides. If a polygon could be divided into as many triangles as it has sides, then the sum of all its angles would evidently be equal to twice as many right angles as there are sides (§ **64**). But since by the last section it appears that the number of these triangles is *two* less than the number of sides, it follows that the sum of all the angles will be twice as many right angles as there are sides less two. Hence, —

Theorem. — *The sum of the angles of a polygon is equal to twice as many right angles as there are sides less two.*

Thus the hexagon (*Fig. 108*) consists of $6 - 2 = 4$ triangles; and the sum of its angles $= 4 \times 2 = 8$ right angles $= 720°$.

In general, if n be the number of sides (n being any number), the following equation or FORMULA enables us to find the sum of the angles: —

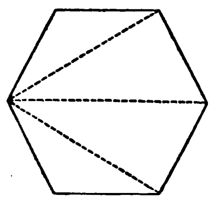


Fig. 108.

Sum of the angles (in degrees) = 180(n − 2). [1.]

Exercises. — **1.** What polygon cannot have obtuse angles?

2. Find the sum of the angles of polygons of 3, 4, 5, 6, 8, 10, 12, 20 sides.

3. Find by division into triangles the sum of the angles of the polygon *A* (*Fig. 107*). Also find the sum by means of Formula [1].

4. Find in the same two ways the sum of the angles of the polygon B (*Fig. 107*).

5. Of how few sides can you make a polygon with one reëntrant angle? with two reëntrant angles?

6. Make a hexagon with as many reëntrant angles as possible.

7. Make a hexagon having only right angles or their multiples.

8. Draw a polygon such that one of its diagonals shall lie wholly outside the polygon.

§ **117**. **Problem.** — *To make a polygon equal to a given polygon ABCDEF (Fig. 109).*

Analysis. — Divide the given polygon into triangles by drawing diagonals from one corner; then construct a series of new triangles equal respectively to those of the given polygon, and *similarly placed.* It is evident that these taken together will form a new polygon equal to the given polygon (Axiom II.).

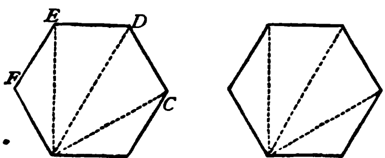


Fig. 109.

Draw the figures and give the construction in full.

Exercises. — **1.** Construct a pentagon equal to a given pentagon.

2. Construct an octagon equal to a given octagon.

3. When are two polygons equal to each other?

II.—Regular Polygons.

§ **118**. **Definitions.** — *A polygon which has equal sides is termed* EQUILATERAL (*Fig. 110, A*); *a polygon which has equal angles is termed* EQUIANGULAR (*Fig. 110, B*); *and a polygon which has equal sides and equal angles* (*Fig. 110, C*) *is termed* REGULAR.

Since the angles of a regular polygon are equal, each angle is equal to *the sum of all the angles divided by their number; or, in other words, divided by the number of sides of the polygon.*

If we let n denote the number of sides, the sum of the angles

(by § **116**) = 180 (n − 2) = 180 n − 360. Dividing this value by n we obtain (Axiom V.) the formula,—

$$\textbf{Each angle of a regular polygon} = 180 - \frac{360}{n}. \qquad [2.]$$

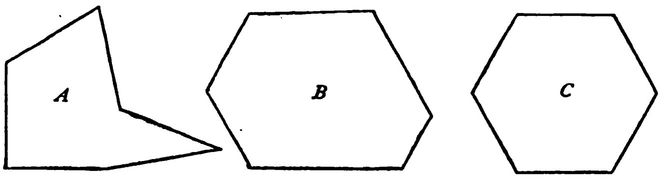


Fig. 110.

Exercises.—1. What is the regular triangle called? the regular quadrilateral?

2. Can a triangle be equilateral without being equiangular?

3. What quadrilateral is equilateral but not equiangular? equiangular but not equilateral?

4. Make three hexagons: the first equilateral but not equiangular; the second equiangular but not equilateral; the third regular.

5. Find each angle of a regular polygon of 3, 4, 5, 6, 8, 10, 12, 20 sides, and arrange the results in a vertical column. As the number of sides increases how does the angle change?

6. Find the angle of a regular polygon of 360 sides.

7. Express Formula [2] in words.

8. One side of a regular dodecagon is 18^m; find its perimeter. Give a rule for finding the perimeter of a regular polygon when a side is known.

§ **119**. Regular polygons possess a very important property stated in the following theorem:—

Theorem.—*The bisectors of the angles of a regular polygon meet in a point equally distant* (i.) *from all the corners, and also* (ii.) *from all the sides.*

Proof.—(*i.*) Let AO (*Fig. 111* or *Fig. 112*) bisect the angle A, BO the angle B, CO the angle C, etc. The triangles AOB, BOC, etc., are isosceles (§ **81**); they are also equal (I. Law of Equality);

whence it follows that the bisectors must meet, as we have assumed, in a common point O, and that this point is equally distant from all the corners. In other words, the distances OA, OB, OC, etc., are all equal.

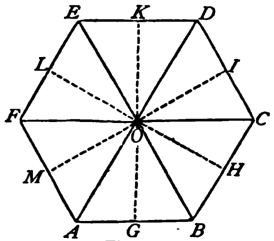


Fig. 111.

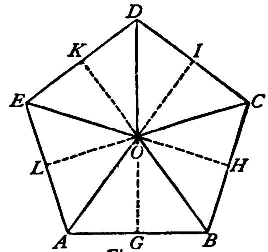


Fig. 112.

(*ii.*) The distances from O to the sides of the polygon are the perpendiculars let fall from O to the sides (§ **83**, Corollary). Take any two of these perpendiculars OG and OH; they are equal, because $\triangle\ OGB \cong OHB$ (why?). In like manner we can show that $OH = OI = OK$, etc.; in other words, these perpendiculars are all equal.

Definitions. — *This remarkable point O in a regular polygon is called its* CENTRE; *the distance from the centre to a corner is called the* GREATER RADIUS; *the distance from the centre to a side is called the* LESS RADIUS; *the angle between two radii is called the* ANGLE AT THE CENTRE.

Corollaries. — 1. *Every regular polygon can be divided into as many equal isosceles triangles as it has sides* (how?).

2. *Every regular polygon can be divided into twice as many equal right triangles as it has sides* (how?).

3. *The greater radii of a regular polygon are all equal, the less radii are equal, and the angles at the centre are equal.*

4. *The angle at the centre is equal to 360 divided by the number of sides; in other words, to the last term of* [2], § **118**.

Exercises.—**1.** Prove the preceding theorem for the case of a triangle. Draw a figure and give the proof in full. (Compare this exercise with Exercises 7 and 8, § **92**.)

2. Show that every regular polygon is composed of an *even* number of equal right triangles.

3. Prove that in a regular polygon a radius, if prolonged, divides the polygon into two equal parts.

4. Prove that the angle between two less radii is equal to the angle between two greater radii. What value has the angle between a greater radius and a less radius?

5. Find the angle at the centre in an equilateral triangle? a square? a pentagon? a hexagon? an octagon? a decagon?

6. If the angle at the centre is 10°, how many sides has the polygon, and what is the value of each angle?

7. If an angle at the centre is given, how can you find an angle of the polygon?

8. If an angle of the polygon is given, how can you find an angle at the centre?

NOTE.—For the sake of reference, the chief data respecting the angles of regular polygons are collected in the following table:—

NUMBER OF SIDES.	SUM OF ANGLES.	EACH ANGLE.	ANGLE AT THE CENTRE.
3	180°	60°	120°
4	360°	90°	90°
5	540°	108°	72°
6	720°	120°	60°
8	1080°	135°	45°
10	1440°	144°	36°
12	1800°	150°	30°
20	3240°	162°	18°

§ **120. Problem.**—*To construct a regular polygon having given the number of sides.*

Construction.—Construct about a point as many angles as the polygon has sides, each angle equal to the angle at the centre (how is this angle found?); lay off equal lengths on the sides of these angles, and join the points thus determined.

Exercises. — 1. Is the above problem *determinate* or *indeterminate?*

2. Construct (*i.*) an equilateral triangle; (*ii.*) a square.

3. Construct a regular hexagon. Join the alternate corners by straight lines. What figure do these lines enclose? Why?

4. Prove that the greater radius of a regular hexagon is equal to a side of the hexagon.

5. Construct a regular octagon. Join the alternate corners by straight lines. What figure do these lines enclose? Why?

6. Construct a regular nonagon.

7. Can you devise another way to construct a regular polygon?

§ **121.** **Problem.** — *To find the centre of a given regular polygon.*

Analysis. — Two methods of solving this problem are suggested by § **118**; what are they?

A third method is based on the fact (see *Figs. 111* and *112*) that, if the polygon has an *even* number of sides, a line through one corner and the centre passes through the opposite corner; while if it has an *odd* number of sides, this line passes through the *middle* of the opposite side. Hence, state rules for proceeding in each case.

Exercises. — 1. Find the centre of (*i.*) an equilateral triangle; (*ii.*) a square; (*iii.*) a regular pentagon; (*iv.*) a regular hexagon.

2. Divide an equilateral triangle into 3 equal parts.

3. Divide a square into 4 equal parts. Can you do this in more than one way?

4. Divide a regular hexagon into 6 equal parts; also into 12 equal parts.

5. Divide a regular octagon into 4, 8, 16 equal parts.

NOTE. — The construction, etc., of regular polygons will be considered further in the chapter on the Circle.

§ **122.** **Applications.** — The stone pavements of large vestibules, galleries, etc., and the patterns used in quilting, are illustrations of the practical employment of regular polygons.

Not every regular polygon can be employed for these purposes; only those can be used whose sides touch at all points; for, if this

were not the case, the surface would not be everywhere covered. We meet most commonly with groups of squares, hexagons, or octagons. Let us examine some cases.

1. Equilateral triangles (*Fig. 113*) can be used. For, since each angle is equal to 60°, it is obvious that the triangles may be arranged in groups of six about a point, each group constituting a regular hexagon.

2. The arrangement in squares (like a checker-board) is shown in *Fig. 114*.

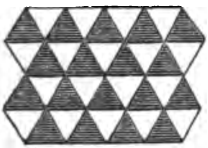


Fig. 113.

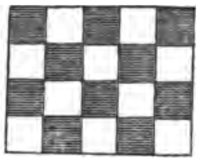


Fig. 114.

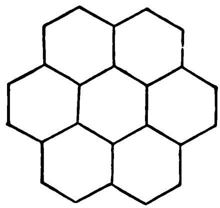


Fig. 115.

3. Regular hexagons (*Fig. 115*) may be arranged in groups of three about a point; for the angle of the hexagon being 120°, three angles $= 3 \times 120° = 360°$.

4. *Fig. 116* shows a pattern consisting of hexagons and equilateral triangles combined.

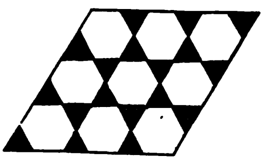


Fig. 116.

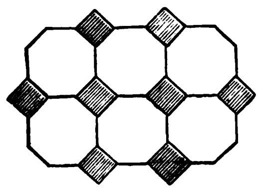


Fig. 117.

5. Regular octagons alone are not sufficient; for the angle of the octagon is 135°, and twice 135° is only 270°, and there remains an angle of 90° to be filled. This can be done by the right angle of a square (*Fig. 117*).

6. The angle of a regular decagon is 144°. Two decagons placed with one side common would leave an angle of $360° - 2 \times 144 = 72°$ to be filled. Since no regular polygon has an angle of 72°, it follows that we cannot pave or quilt with regular decagons, either alone or combined with other regular polygons.

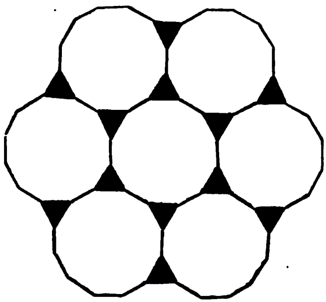


Fig. 118.

7. The regular dodecagon may be combined with the equilateral triangle; for the angle of the dodecagon is 150°, twice 150° is 300°, and there remains 60°, which is just equal to the angle of the equilateral triangle (*Fig. 118*).

8. Thus far we have spoken only of regular polygons. It is obvious that suitable combinations may also be made with polygons which are not regular. *Fig. 119* gives an example. What is the name of the polygon?

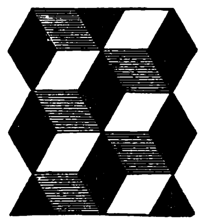


Fig. 119.

Exercise. — Show that a square, a hexagon, and a dodecagon will fill up the space about a point; and make a pattern of these polygons.

III. — Symmetrical Figures.

§ **123.** If the polygon $ABCD$ is made to revolve about AD through half a revolution, the figure $AEFD$, thereby obtained, is said to be SYMMETRICAL with respect to $ABCD$, and AD is called the LINE OF SYMMETRY.

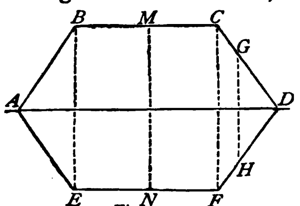


Fig. 120.

Two symmetrical plane figures are always equal, but their equal parts (sides and angles) follow in opposite order with reference to the line of symmetry.

From what precedes it is obvious that in two symmetrical plane figures, —

I. *Two symmetrically-situated sides are equal.*

II. *Two symmetrically-situated angles are equal.*

III. *The line joining two symmetrically-situated points is perpendicular to the line of symmetry and bisected by it (for example, the lines BE, MN, CF, etc., Fig. 120).*

By means of III. we can easily construct upon one side of a polygon, as a line of symmetry, a symmetrical polygon. Explain how.

If we regard $ABCDFE$ as one polygon, then AD is a line of symmetry dividing it into two symmetrical polygons.

A line may divide a figure into parts which are equal and yet not symmetrical. This is the case, for example, with the diagonal of the parallelogram $ABCD$ (*Fig. 94*). It divides the figure into two equal triangles, but these triangles are not so placed with respect to the diagonal, that they would coincide if folded over the diagonal as an edge.

A figure may have more than one line of symmetry. Thus, in *Fig. 120*, besides AD, the line MN is also a line of symmetry.

Exercises.—**1.** Point out in *Fig. 120* symmetrically-situated lines and angles.

2. Draw a parallelogram, and upon one side construct a symmetrical parallelogram.

3. Construct upon one side of a hexagon a symmetrical hexagon.

4. Construct two symmetrical figures with reëntrant angles.

5. Draw a line of symmetry in an isosceles triangle.

6. Draw all the lines of symmetry in an equilateral triangle.

7. Draw all the lines of symmetry in a square.

8. Draw all the lines of symmetry in a rectangle.

9. Make a trapezoid such that you can draw in it a line of symmetry.

10. Give instances of figures which, placed together with a side common, will make a symmetrical figure. What is the line of symmetry?

11. Draw all the lines of symmetry in a regular hexagon. How many are there?

REVIEW OF CHAPTER VI.

SYNOPSIS.

1. A ***polygon*** is a plane figure bounded by straight lines.

2. Polygons receive different names according to the number of their sides.

3. The sum of the angles of a polygon = 180° × the number of sides less two.

4. Two polygons are equal, if they can be divided into the same number of triangles, similar each to each, and similarly placed.

5. A polygon is ***equilateral***, if its sides are equal; ***equiangular***, if its angles are equal; ***regular***, if both sides and angles are equal.

6. Each angle of a regular polygon may be found by dividing 360° by the number of sides, and then subtracting the quotient from 180°.

7. In every regular polygon there is a point called the ***centre***, equally distant from its corners and also from its sides.

8. The ***greater radius*** of a regular polygon is the distance from the centre to a corner; the ***less radius*** the distance from the centre to a side; the ***angle at the centre*** the angle included between two greater radii.

9. Paving and quilting furnish examples of the practical use of regular polygons.

10. Two plane figures are ***symmetrical***, if either of them, when revolved about a certain line as an axis through half a revolution, can be made to fall upon and coincide with the other. The line is called the ***line of symmetry***.

11. There are figures in which several lines of symmetry may be drawn, each dividing the figure into two symmetrical figures.

EXERCISES.

1. Make a hexagon having four right angles.

2. Make an octagon having only right angles or their multiples.

3. Make three polygons with reëntrant angles.

4. Divide a hexagon into other figures in several different ways.

5. Find the sum of the angles of a polygon with 24 sides.

6. Find each angle of a regular icosagon.

7. Eight angles of a decagon are equal, and each of the other two is twice one of the former; find all the angles.

8. In a dodecagon each angle taken in order is $1^\circ 20'$ more than the preceding; find all the angles.
9. Find the angles of a pentagon, if they are to each other as the numbers 1, 2, 3, 4, 5.
10. Draw a heptagon; then construct another equal to it.
11. Construct a regular decagon.
12. Make a five-rayed star, and prove that the sum of the five acute angles is equal to 180°.

    Hint. — The star is made by prolonging the sides of a regular pentagon till they meet, and then erasing the sides of the pentagon.
13. Make a seven-rayed star, and find the sum of the seven acute angles.
14. Can you make a pattern of squares, pentagons, and icosagons?
15. Can you make a pattern of pentagons and decagons?
16. Construct a triangle, and upon its longest side a symmetrical triangle.
17. Construct a rhombus; then construct upon one of its sides a symmetrical rhombus.
18. Draw a polygon with two reëntrant angles, and then construct a symmetrical polygon.
19. Make a right triangle in which a line of symmetry can be drawn.
20. How many lines of symmetry can be drawn in a regular polygon of any number (n) of sides?

CHAPTER VII.

AREAS.

CONTENTS.— I. Units of Area (§§ 124, 125). II. The Areas of Polygons (§§ 126–135). III. Practical Exercises and Applications (§§ 136–141). IV. Theorem of Pythagoras (§§ 142, 143). V. Transformation of Figures (§§ 144–150). VI. Partition of Figures (§§ 151–155).

I.—Units of Area.

§ **124.** Surfaces, like lines, are measured by choosing a unit, and then finding how often this unit is contained in the surface which we wish to measure.

The unit chosen for this purpose must be a magnitude of the same kind as that which is to be measured; in other words, it must be a surface. The most convenient units to employ are *squares whose sides are equal to the units of length.*

Chief English units: the SQUARE INCH (sq. in.), the SQUARE FOOT (sq. ft.), the SQUARE YARD (sq. yd.), and the SQUARE MILE. $30\frac{1}{4}$ sq. yds. = 1 SQUARE ROD, and 160 sq. rods = 1 ACRE.

The metric units (with their abbreviations[1] and relative values) are as follows: —

SQUARE KILOMETER (qkm)	=	1,000,000	square meters.
SQUARE HECTOMETER (qhm)	=	10,000	" "
SQUARE DEKAMETER (qdkm)	=	100	" "
SQUARE METER (qm)	=	1	" "
SQUARE DECIMETER (qdm)	=	0.01	" "
SQUARE CENTIMETER (qcm)	=	0.0001	" "
SQUARE MILLIMETER (qmm)	=	0.000001	" "

[1] Formed by prefixing *q*, the first letter of the Latin word *quadra* (a square) to the corresponding linear abbreviations.

The square dekameter is usually called an AR (a), and the square hectometer a HECTAR (ha). They are employed chiefly in measuring land.

NOTE.—$1^{ha} = 2.471$ acres; $1^{qm} = 10.764$ sq. ft.; $1^{qdm} = 15.5$ sq. in., very nearly.

§ **125.** In the table of metric units above given, it will be observed that each unit is 100 times greater than the unit next following. The reason of this lies in the fact that, of the corresponding linear units, one is 10 times greater than the other (see page 41).

Suppose that AB (*Fig. 121*) represents a length of one meter. Construct upon AB the square $ABCD$; then this square will represent one square meter. Divide AB and AC each into ten equal parts, and through all the points of division draw lines parallel to the adjacent sides. The parallels through the points of division of AB divide the square into 10 equal rectangulars, each 10^{dm} long and 1^{dm} wide; the other set of parallels subdivides each rectangle into 10 equal squares, each equal to one square decimeter; therefore, one square meter must contain exactly $10 \times 10 = 100$ square decimeters.

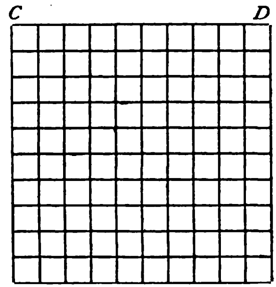


Fig. 121.

The same reasoning will show that one square decimeter $= 10 \times 10 = 100$ square centimeters, etc.; and if the ratio of the two linear units were any other number but 10, it may be shown in the same way that the ratio of the corresponding units of surface would be found by multiplying that number by itself; in other words, by *squaring* it.

The ratio of two units of surface is always the square of the ratio of the corresponding units of length.

Exercises.—**1.** What is the ratio of a square foot to a square inch? a square foot to a square yard? a square mile to a square foot?

2. How many square feet are there in one acre?

3. How many acres are there in one square mile?

4. What is the ratio of an ar to a square meter? an ar to a hectar? a square centimeter to a hectar? a square hectometer to a hectar?

5. Reduce to square meters the following: 4^{qkm}; 15^{ha}; 9.87^{a}; 0.084^{ha}; $97{,}675^{qcm}$.

6. Reduce to hectars the following: 63^{qkm}; 39.2^{a}; 3600^{qm}; $56{,}475^{qdm}$; $4{,}000{,}000^{qmm}$.

7. Subtract 1^{a} from 1^{ha}, and give the answer in square meters.

8. How many house-lots, each containing 2^{a}, are there in a field which contains 8^{ha}?

9. A man bought 3^{ha} of land at $200 per hectar, and sold it for $2.50 per ar. Did he gain or lose, and how much?

10. Which is the greater farm, 200 hectars or 500 acres?

11. Divide 6^{ha} into 64 equal lots; how many square meters in each lot?

12. What is the difference between 3 *square meters* and 3 *meters square?*

II.—Areas of Polygons.

§ **126. Definition.**—*The number of times a unit of surface is contained in any surface, followed by the name of the unit, is called the* AREA *of the surface.*

NOTE.—Hence units of surface are often called *units of area*, as at the beginning of this chapter.

In order to find the area of a surface, it might at first thought seem necessary to apply a unit of area to the surface over and over again as many times as possible, just as we measure a line directly by using a yard-stick or a meter-rule. But this method would be very tedious, and in many cases (for instance, a pond, swamp, forest, etc.) utterly impracticable.

Fortunately, however, Geometry supplies us with indirect methods of measurement which are applicable to all surfaces. It teaches that the value of areas depends upon the *lengths* of certain lines; whence it follows that areas can be found by performing certain

simple operations upon the numbers which express the lengths of these lines.

In explaining these indirect methods we begin with the square and the rectangle, because they can be readily decomposed into square meters, or into squares larger or smaller than the square meter.

§ **127**. The SQUARE. If the side of a square contains a whole number of meters, as six, it is easy to see that (by proceeding exactly as in § **125**) we can first divide the square into six equal rectangles, then each rectangle into six square meters; so that the square will contain in all $6 \times 6 = 36$ square meters.

The same mode of reasoning may be used when the side contains meters and subdivisions of a meter. Let, for instance, the side $= 6.35^m$; then, in decomposing the square into smaller squares, we may take as the unit of length one centimeter. In place of six rectangles, there will be 635; and in place of $6 \times 6 = 36$ square meters, there will be $635 \times 635 = 403,225$ square centimeters $=$ 40.3225 square meters, — a result obtained simply by multiplying 6.35 by itself.

In short, *the number of units of area in a square is always found by multiplying by itself the number of the corresponding units of length in one of its sides.*

Hence, when we multiply a number by itself, we are said to *square* it; and it is usual to express the preceding rule more briefly thus: *The area of a square is equal to the square of one of its sides.*

Conversely, if the area is known, in order to find the value of one side, we must find a number which, multiplied by itself, will give the numerical value of the area; in other words, we must *extract the square root of the given area.*

For example: if the area of a square $= 8.1225$ square meters, one side $= \sqrt{8.1225} = 2.85$ meters.

Remark. — For the sake of convenience we use the expression

"the square of AB," and we write $\overline{AB}^2$. These expressions do not mean that the *line* AB is multiplied by itself, but that the *number of linear units* in the line is multiplied by itself, the result being the *number of units of area* in the square constructed upon AB as a side.

§ **128.** The RECTANGLE. A rectangle can be decomposed into units of area in the same way as a square, the only difference being that the number of units of length in its adjacent sides is not the same.

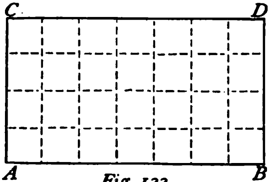


Fig. 122.

Thus, the rectangle $ABCD$ (*Fig. 122*), the sides of which are $AB = 7^m$, $AC = 4^m$, contains $7 \times 4 = 28$ square meters.

The side AB is the base of the rectangle, and the side AC is the altitude (§ **104**); so that *the number of units of area in a rectangle is found by multiplying the number of linear units in the base by the number of linear units in the altitude.*

Or, more briefly: *The area of a rectangle is equal to the product of its base by its altitude.*

If the area and base are known, how can the altitude be found? If the area and altitude are known, how can the base be found?

Remark.—In practice we must be careful to express both base and altitude in terms of the *same* linear unit; the area will then be expressed in terms of the *corresponding* unit of area.

For example: the base of a rectangle $= 12^m$, the altitude $= 80^{cm}$; find the area. Here we must not multiply 12 by 80; but, if we choose the meter as the linear unit, we must multiply 12 by 0.8, because $80^{cm} = 0.8^m$. Answer, 9.6 square meters.

If we take the centimeter as the linear unit, what will the answer be?

§ **129.** The PARALLELOGRAM. It is easy to transform a parallelogram into an equivalent rectangle.

In the parallelogram $ABCD$ (*Fig. 123*) erect perpendiculars at the points A and B of the base; they cut, one of them the opposite side, the other the opposite side prolonged, in the points F and E; and the figure $ABEF$ is a rectangle having the same base and altitude as the parallelogram.

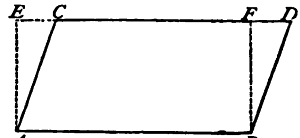


Fig. 123.

The right triangles ACE and BDF are equal (why?); hence, whichever of these triangles we add to the quadrilateral $ABCF$, we shall obtain the same area. If we add $\triangle ACE$, we obtain the rectangle $ABEF$; and if we add $\triangle BFD$, we obtain the parallelogram $ABCD$: therefore, the parallelogram $ABCD$ is equal to the rectangle $ABEF$.

Theorem.—*A parallelogram is equivalent to a rectangle having the same base and altitude.*

NOTE.—This transformation may be easily shown to the eye by cutting from stiff paper the trapezoid $ACBF$ (*Fig. 123*), and the right triangle AEC, and then placing the triangle in the two positions shown in the figure.

Corollaries.—1. By combining this theorem with § **128**, we see that *the area of a parallelogram is equal to the product of its base by its altitude.*

2. *All parallelograms having equal bases and equal altitudes are equivalent.*

Is the converse of Corollary 2 true? Draw a diagram to illustrate.

Construct several equivalent parallelograms differing in shape.

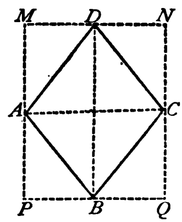


Fig. 124.

Remark.—In a *rhombus* the diagonals bisect each other at right angles (§ **104**).

Construct about the rhombus $ABCD$ (*Fig. 124*) a rectangle $PQMN$, by drawing through the corners of the

rhombus lines parallel to its diagonals. It is easy to show that this rectangle is composed of eight equal right triangles, of which the rhombus contains four (how?). Now, since the base and altitude of the rectangle are equal to the two diagonals of the rhombus, its area is equal to the product of these diagonals. Hence, *the area of the rhombus is equal to half the product of its diagonals.*

§ **130.** The TRIANGLE. A triangle ABC (*Fig. 125*) may be regarded as half a parallelogram having an equal base and equal altitude (§ **101**). We have only to draw through two corners, B and C, of the triangle lines parallel to the opposite sides; then, since $\triangle ABC = \frac{1}{2} ABDC$, and $ABDC = AB \times CE$, therefore $\triangle ABC = \frac{1}{2} AB \times CE$. That is,—

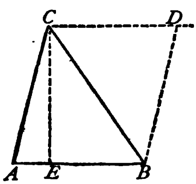


Fig. 125.

The area of a triangle is equal to half the product of its base by its altitude.

Corollary.—*All triangles having equal bases and equal altitudes are equivalent.*

NOTE.—In finding the area of a triangle, we may either multiply the base by the altitude, and then divide by 2, or we may multiply half the base by the altitude, or we may multiply half the altitude by the base. For example: if the base $= 14^m$, and the altitude $= 8^m$, the area $= \frac{14 \times 8}{2} = \frac{14}{2} \times 8 = 14 \times \frac{8}{2} = 56$ square meters.

§ **131.** The TRAPEZOID. A trapezoid $ABCD$ (*Fig. 126*) is divided by the diagonal BC into two triangles, ABC and BCD, whose bases, AB and CD, are the parallel sides of the trapezoid, and whose common altitude CE is the altitude of the trapezoid.

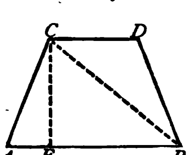


Fig. 126.

$$\text{Since } \triangle ABC = \tfrac{1}{2} AB \times CE,$$
$$\text{and } \triangle BCD = \tfrac{1}{2} CD \times CE;$$

therefore, the trapezoid $ABCD = \frac{1}{2}(AB + CD) \times CE$.

That is, *the area of a trapezoid is equal to half the product of the sum of the parallel sides by the altitude.*

§ **132**. Any Quadrilateral. To find the area of any quadrilateral, *divide it by a diagonal into two triangles, find the areas of these triangles, and add the results together.*

§ **133**. A Regular Polygon. We know (§ **119**, Corollary 1) that a regular polygon $ABCDEF$ (*Fig. 127*) is composed of as many equal isosceles triangles as there are sides. Therefore its area may be found by adding together the areas of these triangles. The altitude of each triangle is equal to the less radius of the polygon, so that the area of each triangle is half the product of its base by the less radius. Therefore the area of the polygon is half the product of the sum of all the bases (that is to say, the perimeter of the polygon) by the common altitude (that is to say, the less radius of the polygon).

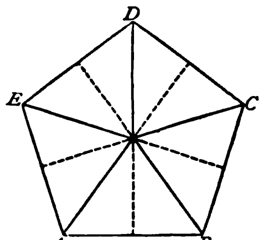


Fig. 127.

The area of a regular polygon is half the product of its perimeter by its less radius.

§ **134**. Any Polygon. The area of any polygon may be found by dividing it into triangles. *Fig. 128* shows two ways of doing this. In either case, if we compute the area of each triangle separately, and then add the results, we shall find the area of the polygon.

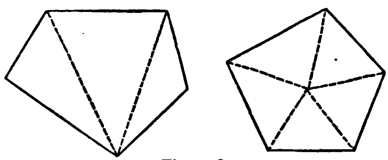


Fig. 128.

In each of these cases how many lines require to be measured *directly* in order to compute the area?

But it is generally easier to find the area of an irregularly-shaped

polygon (as a field, farm, etc.) by division into right triangles and trapezoids, as shown in *Fig. 129.* Join the two most distant corners of the polygon by a straight line, and let fall from the other corners perpendiculars to this line taken as a common base. The polygon is thus divided into right triangles and trapezoids, the areas of which are then computed separately, and their sum is obviously equal to the area of the polygon.

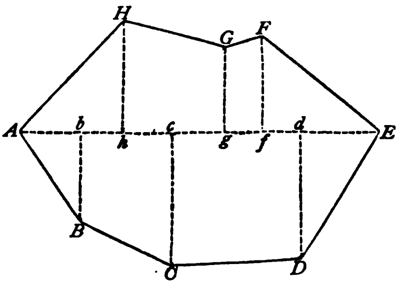


Fig. 129.

For example: let (*Fig. 129*) $Bb = 6.8^{m}$; $Cc = 10.6^{m}$; $Dd = 10.1^{m}$; $Ff = 8.3^{m}$; $Gg = 6.2^{m}$; $Hh = 9.2^{m}$: also let $Ab = 5.6^{m}$; $bh = 2.6^{m}$; $hc = 4.2^{m}$; $cg = 4.6^{m}$; $gf = 3^{m}$; $fd = 2.8^{m}$; $dE = 5.8^{m}$.

The work of computation may be arranged as follows: —

PARTS OF THE POLYGON.	FACTORS.		PRODUCTS.
	BASES, OR SUMS OF PARALLEL SIDES.	ALTITUDES.	
$\triangle ABb$	$Bb = 6.8$	$Ab = 5.6$	38.08
Trap. $BbcC$	$Bb + Cc = 17.4$	$bc = 6.8$	118.32
Trap. $CcDd$	$Cc + Dd = 20.7$	$cd = 10.4$	215.28
$\triangle DdE$	$Dd = 10.1$	$dE = 5.8$	58.58
$\triangle FfE$	$Ff = 8.3$	$fE = 8.6$	71.38
Trap. $FfgG$	$Ff + Gg = 14.5$	$fg = 3$	43.50
Trap. $GghH$	$Gg + Hh = 15.4$	$gh = 8.8$	135.52
$\triangle AhH$	$Ah = 8.2$	$Hh = 9.2$	75.44
			2)756.10
	Polygon $ABCDEFGH =$		378.05^{qm}

NOTE 1.—In the preceding table, in place of dividing each product by 2, we first add the products, and then divide their sum by 2.

NOTE 2.—In case the interior of the polygon is inaccessible, *Fig. 130* illustrates one way of meeting the difficulty. Construct a rectangle which shall completely enclose the polygon, and, from all the corners of the polygon which are not in the

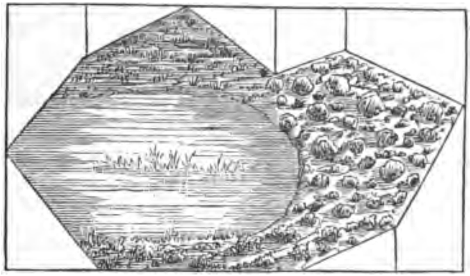


Fig. 130.

sides of the rectangle, let fall perpendiculars to the sides of the rectangle. In this way a number of right triangles and trapezoids are formed, the areas of which can be found in the usual way; and it is evident that if the sum of their areas is subtracted from the area of the rectangle, the remainder will be equal to the area of the polygon.

§ **135.** It will now be clear in what way Geometry supplies us with indirect methods of finding areas. In the first place, it teaches that the area of a rectangle (the square is simply a rectangle with equal sides) depends entirely on the *lengths* of its two sides, and can therefore be found as soon as the lengths of these sides are known; it then shows that any parallelogram can be transformed into an equivalent rectangle, that a triangle is half a parallelogram of the same base and altitude, and that a trapezoid consists of two triangles of equal altitude; lastly, it shows that any polygon whatsoever may be decomposed into triangles and trapezoids.

Remark.—In some cases the rule for finding an area may be very briefly expressed by a formula in which letters are used to represent the values of the various quantities.

Let S denote in all cases the area, a the side of a square, a and b the adjacent sides of a rectangle, a and h the base and altitude

of a parallelogram or triangle, a and c the parallel sides, h the altitude of a trapezoid, p and r the perimeter and less radius of a regular polygon. Then, —

For the square,	$\mathbf{S = a^2}$,	[3.]
For the rectangle,	$\mathbf{S = a \times b}$,	[4.]
For the parallelogram,	$\mathbf{S = a \times h}$,	[5.]
For the triangle,	$\mathbf{S = \frac{1}{2} a \times h}$,	[6.]
For the trapezoid,	$\mathbf{S = \frac{1}{2} (a + c) \times h}$,	[7.]
For the regular polygon,	$\mathbf{S = \frac{1}{2} p \times r}$,	[8.]

III.—Practical Exercises and Applications.

NOTE.—In solving most of the exercises, it is well to draw (free-hand) a figure representing as nearly as possible the true shape of the polygon.

§ 136. THE SQUARE ($S = a^2$).

In a square, given, —

1. The side $= 23^{m}$; find the area.

2. The side $= 35^{m}$; find the area.

3. The side $= 4\frac{1}{2}^{ft}$; find the area.

4. The side $= 5\frac{3}{4}^{yds}$; find the area.

5. The side $= 6.25^{m}$; find the area.

6. The side $= 48.72^{m}$; find the area.

7. The area $= 625^{qm}$; find the side.

8. The area $= 515.29^{qm}$; find the side.

9. The area $= 63.2025^{qm}$; find the side.

10. The area $= 54.76^{a}$; find the side in meters.

11. The area $= 16.3216^{ha}$; find the side in meters.

12. The area $= 78.8544^{ha}$; find the side in dekameters.

13. One side of a square garden measures 37.6^{m}; find its area.

14. A square park contains 64 acres. How many trees, 20^{yds} apart, can be set out around it?

15. A square park contains 64^{ha}. How many trees, 20^{m} apart, can be set out around it?

16. The three dimensions of a room are each equal to 6.4^{m}. What will it cost to paper its walls, if the price is \$0.15 per square meter?

17. Find the cost of 1000 panes of glass, each 0.48^{m} square, if the glass is worth \$1.40 per square meter.

18. Area of the State of New York = 47,156 square miles. Find the side of a square having the same area.

19. Construct a square, and then find its area.

§ 137. The Rectangle ($S = a \times b$).

In a rectangle, given,—

20. Length = 60^{m}, breadth = 24^{m}; find the area.

21. Length = 2^{m}, breadth = 80^{cm}; find the area.

22. Length = 8.5^{kil}, breadth = 40^{m}; find the area.

23. Length = 92^{dkm}, breadth = 16.37^{dkm}; find the area in hectars.

24. Length = $28\frac{3}{4}^{ft}$, breadth = $17\frac{1}{2}^{ft}$; find the area in sq. yds.

25. Length = 627^{yds}, breadth = 435^{yds}; find the area in acres.

26. Area = 360^{qm}, length = 30^{m}; find the breadth.

27. Area = 5.6^{a}, length = 16^{m}; find the breadth.

28. Area = 50^{ha}, length = $\frac{5}{8}^{km}$; find the breadth.

29. Area = 6 acres, breadth = 44^{yds}; find the length.

30. Area = 767.24^{qm}, breadth = 24.5^{m}; find the length.

31. Area = 7.2^{ha}, breadth = 0.25^{kil}; find the length,

32. Find the area of the surface of a table 1.58^{m} long, 1.2^{m} wide.

33. How many ars in a field 87.5^{m} by 45.8^{m}?

34. Find the area of a field 240^{m} long and $\frac{5}{12}$ as wide.

35. The perimeter of a rectangle = 24^{m}, the length = 9.2^{m}; find the breadth and the area.

36. If the area of a cellar 48^{m} long is 1500^{qm}, what is its breadth?

37. The perimeter of a rectangle = 86.2^{m}, the breadth = 12.6^{m}; find the length and the area.

38. The area of a rectangle = 144^{qm}, the breadth = 6^{m}; find the length and the perimeter.

39. The area of a rectangle = S, the length = a; find the breadth and the perimeter.

40. Construct a rectangle, and then find its area.

41. Find the area of the floor of the room.

42. The perimeter of a rectangle = 72^{m}, and the length is twice the breadth; find length, breadth, and area.

43. If the perimeter = 96^{m}, and the length is to the breadth as 7 : 5, find length, breadth, and area.

44. If the perimeter = 26^{m}, and the length is 2.5^{m} more than the breadth, find length, breadth, and area.

45. How much larger is a rectangle if the length is doubled? if the breadth is doubled? if both are doubled?

46. The sides of a rectangle are 42^m and 60^m; find the side of a square equivalent to the rectangle.

47. If the side of a square $= 12^m$, find the sides of an equivalent rectangle. Is there any thing peculiar about this exercise?

48. A square and a rectangle have the same perimeter, 60^m, and the rectangle is twice as long as it is wide. Find their areas.

49. Construct a rectangle 16^{dm} long, 4^{dm} wide; then another, 15^{dm}, 5^{dm} wide; then another, 14^{dm}, 6^{dm} wide; and so on, until you reach a square. Then compare perimeters and areas. Which has the greatest area?

50. A chain 80^m long is made to enclose a rectangle, one side being 25^m. How much more area would it enclose if the figure were a square?

51. A and B exchange fields of equally good soil. A's field is rectangular in shape, 4 times as long as wide, and with a perimeter $= 1000^m$; B's field is in the shape of a square, the perimeter being 800^m. Which has the best of the bargain?

52. From a rectangular field 528^m by 240^m a piece 60^m square is sold. How much is left?

Find the number of linear meters of carpeting required for floors of the following dimensions: —

53. Floor, 8^m by 6.6^m; carpet, 1^m wide.

54. Floor, 9.5^m by 7.7^m; carpet, 80^{cm} wide.

55. Floor, 12^m by 8.4^m; carpet, 1.2^m wide.

56. Floor, 24^{ft} by $16^{ft}\ 8^{in}$; carpet, 30^{in} wide.

Find the cost of carpeting rooms, their dimensions and the width and price of carpeting being as follows: —

57. Floor, 10^m by 6.5^m; carpet, 90^{cm} wide, at \$1.25 per meter.

58. Floor, 11.4^m by 5^m; carpet, 1.4^m wide, at \$2 per meter.

59. Floor, 8^m by 7.37^m; carpet, 60^{cm} wide, at \$0.75 per meter.

60. Floor, $18\frac{1}{2}^{ft}$ by $14\frac{3}{4}^{ft}$; carpet, $\frac{7}{8}^{yd}$ wide, at \$1.60 per yard.

61. How much oil-cloth will it take to line the inside of a box 2.6^m by 1.2^m by 0.8^m?

62. How many tiles 8^{in} square will cover a floor 36^{ft} by 20^{ft}?

63. What will it cost to paint the floor and walls of a room 8^m long, 6^m wide, 3^m high, at one cent per square centimeter?

64. Find the cost of glazing 12 windows, each $6^{ft}\ 6^{in}$ by $3^{ft}\ 4^{in}$, at \$0.50 per square foot?

65. A room is 10^m long, 6^m wide, and 3.6^m high. There are four windows,

each 2.2^m by 1.2^m; two doors, each 3^m by 1.8^m; and the base-board is 90^{cm} high. Required (*i.*) the cost of plastering its walls and ceiling, at \$0.40 per square meter; (*ii.*) the cost of papering its walls with paper 80^{cm} wide and 10^m in a roll, at \$1.75 a roll put on, and with a gilt border next the ceiling that costs \$0.20 per meter.

66. A man has a rectangular garden 68^m long and 42^m wide. He makes around it a path 3.4^m wide. What area is left for other purposes?

67. A field is 600^m by 240^m. How much of its area will be taken by a road 20^m wide passing through the middle and parallel to the longer side?

68. Through the middle of a rectangular garden 180^m by 76^m, two paths at right angles to each other and parallel to the sides are made, each 1.4^m wide. Find the area (*i.*) of the paths; (*ii.*) of what remains.

69. A mirror is 2.8^m by 1.9^m, and the wooden frame is 4^{cm} wide. Find the area of the glass and also of the frame.

70. A table 2.3^m by 1.2^m is covered with cloth, which overlaps 3^{cm} on all sides. How much cloth is needed?

71. The bridge over the Elbe at Dresden is 400^m long and 10.3^m wide. How many stones 18^{cm} by 15^{cm} are required to cover the roadway, the foot-paths having together a width of 2.45^m?

72. How many shingles 8^{in} wide, laid 6^{in} to the weather, will cover the roof of a house, the ridge being 52^{ft} long, the slant side of the roof 24^{ft}, and the bottom course on each side being double?

§ 138. THE (oblique) PARALLELOGRAM ($S = a \times h$).

In a parallelogram, given, —

73. Base $= 27^{cm}$, altitude $= 22^{cm}$; find the area.

74. Base $= 91.25^m$, altitude $= 40^m$; find the area.

75. Base $= 640^{dkm}$, altitude $= 180^{dkm}$; find the area in hectars.

76. Area $= 84^a$, base $= 105^m$; find the altitude.

77. Area $= 1^{ha}$, altitude $= 125^m$; find the base.

78. How far apart are the sides of a rhomboid containing 210^{qm}, if one of these sides $= 15^m$?

79. Perimeter of a rhombus $= 6^m$, distance apart of two parallel sides $= 75^{cm}$; find the area.

80. Find the area of a rhombus whose diagonals are 16^m and 12^m.

81. The area of a rhombus $= 240^{qm}$, one diagonal $= 10^m$; find the other.

82. Find the area of a rhombus if the sum of the diagonals $= 20^m$ and their ratio is $3:5$.

83. Construct a rhomboid, and then find its area.

84. Construct a rhombus, and then find its area.

85. Construct a rhombus and a square, both having the same perimeter. Which has the greater area? Find the difference of their areas.

86. A rhombus and a rectangle have equal perimeters. A side of the rhombus $= 5^m$, and a side of the rectangle $= 3.5^m$. Find the perimeter and the two areas.

§ 139. The Triangle ($S = \frac{1}{2} a \times h$).

In a triangle, given,—

87. Base $= 50^m$, altitude $= 40^m$; find the area.

88. Base $= 80.5^m$, altitude $= 60^m$; find the area.

89. Base $= 2.4^{km}$, altitude $= 1.875^{km}$; find the area in hectars.

90. Area $= 450^{ha}$, base $= 5^{km}$; find the altitude.

91. Area $= 40^{qm}$, altitude $= 12.5^m$; find the base.

In a right triangle, given,—

92. The legs 4^m and 5.6^m; find the area.

93. The legs 34^m and 25^m; find the area in ars.

94. The area $= 15^a$, one leg $= 50^m$; find the other leg.

95. Sum of the legs $= 20^m$, their ratio $= 2:3$; find the area.

96. Sum of the legs $= 70^m$, and one is 6 times the other; find the area.

97. Construct a triangle, and then find its area.

98. If the base of a triangle $= 50^m$, what must be its altitude in order that its area may be the same as that of a rectangle whose sides are 40^m and 25^m?

99. Compare (that is, find the ratio of) the areas of a square, and of a triangle whose base is one side of the square and vertex is in the opposite side. Does it matter *where* in the opposite side the vertex is? Why not?

100. Find the side of a square equivalent to a triangle with the base 90^m and the altitude 20^m.

101. The base of a triangle $= 6^m$, altitude $= 4^m$. Find the base of a triangle with the same altitude and twice as large; 3 times as large; 4 times as large; 10 times as large.

102. Two triangles have the same base, 12^m, and their altitudes are 8^m and 24^m. Find the ratio of their areas.

103. Two triangles are each 15^m high, and their bases are 20^m and 30^m. Find the base of a triangle equivalent to their sum.

104. From a triangle with base $= 15^m$, and altitude $= 8^m$, a triangle with the base $= 6^m$ is cut off by a line drawn from the vertex. What part of the whole triangle is the triangle cut off?

105. Same exercise, making the altitude 12^m instead of 8^m.

106. Find the entire surface of a pyramid, with a square base (see page 4, *Fig. 2*, IV.), if a side of the base $= 2^m$, and the altitude of one of the lateral sides $= 4^m$.

107. The roof of a tower consists of 4 isosceles triangles. The base of each triangle $= 2.2^m$, the altitude $= 5.4^m$. How many square meters of tin are required to cover the roof?

§ 140. THE TRAPEZOID ($S' = \frac{1}{2}[a + c] \times h$), AND THE TRAPEZIUM.

108. Find the area of a trapezoid, if the parallel sides are 20^{cm} and 30^{cm}, the altitude 12^{cm}.

109. Area of a trapezoid $= 369^{qm}$, the parallel sides are 16^m and 25^m. Find the altitude.

110. Area of a trapezoid $= 124.8^{qm}$, altitude $= 6.4^m$, one of the parallel sides $= 12.8^m$; find the other.

111. A garden has the shape of a trapezoid. The parallel sides are 60^m and 32^m, and their distance apart 124^m. Find the side of an equivalent square.

112. Construct a trapezoid, and then find its area.

113. A board is 8.5^m long, 0.45^m wide at one end, and 0.36^m wide at the other. Find its value at $0.50 per square meter.

114. A wall 12^m long is 3.15^m high at one end and 2.7^m at the other. Find its area.

115. The area of a field shaped like a trapezoid $= 15^{ha}$. The parallel sides are 168.7^m and 144^m. How far apart are these sides?

116. A court-yard has the shape of a trapezoid. The parallel sides are 20.4^m and 18.2^m, and their distance apart is 16^m. How many stones, each containing 5^{qdm}, will be required to pave the yard?

117. What will it cost to shingle a trapezoidal roof, the parallel sides being 15.8^m and 11.6^m, their distance apart 6.2^m, and the price for shingling being $1.20 per square meter?

118. A diagonal in a trapezium $= 32^m$, and the altitudes of the triangles made by the diagonal are 18^m and 20^m. Find the area of the trapezium.

119. Construct a trapezium, and then find its area.

120. The area of a four-sided field $= 24^{ha}$. The altitudes of the triangles made by a diagonal are 140^m and 158^m. Find the length of the diagonal.

121. In a quadrilateral, a diagonal $= 40^m$, the altitude of one triangle $= 12^m$, and the altitude of the other triangle is four times as great. Find the area of the quadrilateral.

§ 141. Polygons (if regular, $S = \frac{1}{2} p \times r$).

The less radius r of a regular polygon has a fixed ratio (found by Trigonometry) to the length of one side.

In a regular triangle,	$r =$ one side $\times 0.2887$.
In a regular quadrilateral,	$r =$ one side $\times 0.5000$.
In a regular pentagon,	$r =$ one side $\times 0.6882$.
In a regular hexagon,	$r =$ one side $\times 0.8660$.
In a regular octagon,	$r =$ one side $\times 1.2071$.
In a regular decagon,	$r =$ one side $\times 1.5388$.
In a regular dodecagon,	$r =$ one side $\times 1.8660$.

122. One side of a regular triangle $= 20^{cm}$; find the area.

123. One side of a square $= 40^{cm}$; find the area.

124. One side of a regular pentagon $= 50^{cm}$; find the area.

125. The perimeter of a regular hexagon $= 3^{m}$; find the area.

126. The perimeter of a regular octagon $= 16^{m}$; find the area.

127. The perimeter of a regular decagon $= 200^{m}$; find the area.

128. One side of a regular dodecagon $= 60^{m}$; find the area.

129. The area of a regular hexagon $= 6^{ha}$, the less radius $= 240^{m}$; find one side and the perimeter.

130. The area of a regular octagon $= 160^{qm}$, one side $= 5^{m}$; find the less radius.

131. Construct a regular pentagon, and then find its area.

132. Construct a regular hexagon, and then find its area.

133. Find the side of a square equivalent to a regular hexagon one side of which $= 4^{m}$.

134. Find the entire surface of a hexagonal prism (see page 4, *Fig. 2*, III.), if the altitude $= 1.2^{m}$, and one side of the base $= 40^{cm}$.

135. Required the total area of the hexagons in *Fig. 115*, if a side of each hexagon $= 36^{cm}$.

136. Compute the total area covered by the figures in *Fig. 116*, if a side of each figure $= 1.4^{m}$.

137. Compute the total area covered by the figures in *Fig. 117*, if a side of each figure $= 2^{m}$.

138. The floor of a church is 30^{m} by 15^{m}, and is paved with regular hexagonal stones, alternately white and black, a side of each stone being 12^{cm}. How many stones of each kind were required, two per cent being added to fill the corners?

139. A garden in the form of an irregular hexagon can be divided into the following triangles: —

First triangle:	base = 36.6^m, altitude = 6.6^m.
Second triangle:	base = 42.4^m, altitude = 20^m.
Third triangle:	base = 42.4^m, altitude = 22^m.
Fourth triangle:	base = 28.4^m, altitude = 9.8^m.

How many ars does the garden contain?

140. Construct two equal heptagons; then find the area of the first heptagon by one of the methods illustrated in *Fig. 128*, and the area of the second by the method illustrated in *Fig. 129*. If the results do not agree, what is the reason?

IV. — Theorem of Pythagoras.

§ **142.** If we construct a right triangle (*Fig. 131*), having for its legs three and four units of length, we shall find that the hypotenuse will contain exactly five of these units of length. If we then construct squares upon the three sides of this triangle, and divide these squares into square units, as shown in the figure, it will be evident that the square constructed upon the hypotenuse is just equal to the sum of the squares constructed upon the two legs; for, the former square contains 25 square units, and the latter squares 16 and 9 of these units, respectively.

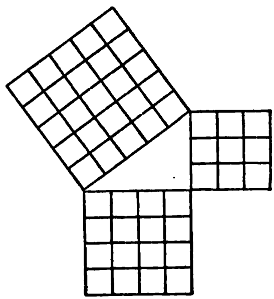


Fig. 131.

This is a simple example of a very useful theorem called, from the name of the discoverer, the Theorem of Pythagoras.[1]

Theorem. — *In every right triangle the square of the hypotenuse is equal to the sum of the squares of the two legs.*

Several different proofs of this theorem have been devised, two of which we shall give.

[1] Pythagoras, born at Samos about 570 B.C., was one of the most renowned philosophers and teachers of antiquity.

Proof I.—Upon the hypotenuse AC (*Fig. 132*) of a right triangle ABC construct the square $ACDE$; from D and E let fall upon AB prolonged the perpendiculars DF and EG; also let fall upon DF the perpendiculars CH and EK. From this construction, it follows that the right triangles ABC, AEG, CDH, and DEK, which taken in order we will call I., II., III., IV., are equal (why?); also, that $BCHF$ is equal to the square upon BC, and $EGFK$ to the square upon AB (why?). Now the square $ACDE$ = the figure $ACHKE$ + the triangles III. and IV.; take from the square these triangles, and place them in the positions of the triangles I. and II., then it is clear that the square will be transformed into the figure $GBCHKE$, which is composed of the squares $BCHF$ and $EGFK$ of the two legs. Hence the square of the hypotenuse $ACDE$ must have the same area as the sum of the squares of the legs.

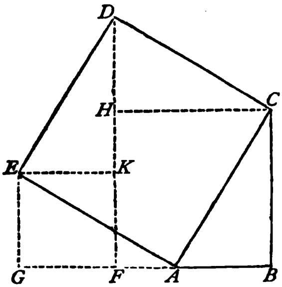


Fig. 132.

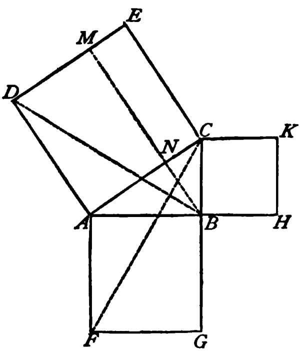


Fig. 133.

NOTE.—This proof can be readily shown in a material form by drawing *Fig. 132* on cardboard, cutting out the triangles I., II., III., and IV., and then changing the places of two of them as just explained.

Proof II. (*Euclid's*[1] *Proof*).—Upon the sides of the right triangle ABC (*Fig. 133*) construct

[1] Euclid was a Greek geometer who lived about 300 B.C. He collected the scattered geometrical knowledge then existing into a famous treatise known as Euclid's Elements,—a book still in use, but as unsuited for beginners as it is admirable for rigor of logic, and for showing on how few axioms the science of Geometry can be made to depend.

squares, and draw $BM \perp AC$. Join BD and CF. $\triangle ABD \cong ACF$ (II. Law of Equality). $\triangle ACF$ has the same base AF and the same altitude AB as the square $ABFG$; therefore (§ **130**) $\triangle ACF = \frac{1}{2}\overline{AB}^2$. And $\triangle ABD$ has the same base AD and the same altitude DM as the rectangle $ADMN$; therefore $\triangle ABD = \frac{1}{2}$ rectangle $ADMN$. But if the halves, here the triangles ACF and ABD, are equal, the wholes must also be equal (Axiom II.); therefore $\overline{AB}^2 =$ rectangle $ADMN$. In the same way (by joining AK and BE), it can be proved that $\overline{BC}^2 =$ rectangle $CEMN$. Therefore $\overline{AB}^2 + \overline{BC}^2 =$ the sum of the rectangles $ADMN$ and $CEMN =$ the square of the hypotenuse AC.

Corollaries.—1. *If from the square of the hypotenuse we subtract the square of either leg, the remainder will be equal to the square of the other leg.*

2. *If a and b denote the legs, c the hypotenuse, of a right triangle,*

$$c^2 = a^2 + b^2, \quad a^2 = c^2 - b^2, \quad \textit{and} \quad b^2 = c^2 - a^2.$$

3. A useful corollary follows from the second proof, namely: *If we construct squares upon the sides of a right triangle, and from the vertex of the right angle draw a line perpendicular to the hypotenuse, this line will divide the square of the hypotenuse into two rectangles, each equal to the square of the adjacent leg.*

§ 143. Practical Exercises and Applications.

In a right triangle, given,—

1. The legs $= 24^m$ and $= 7^m$; find the hypotenuse.
2. The legs $= 35^m$ and $= 12^m$; find the hypotenuse.
3. The legs $= 45^m$ and $= 28^m$; find the hypotenuse.
4. The legs $= 4.7^m$ and $= 1.104^m$; find the hypotenuse.
5. The legs $= 72^m$ and $= 18^m$; find the hypotenuse.
6. The legs $= 2^{dm}$ and $= 15^{cm}$; find the hypotenuse.
7. The hypotenuse $= 65^m$, one leg $= 33^m$; find the other leg.
8. The hypotenuse $= 89^m$, one leg $= 80^m$; find the other leg.
9. The hypotenuse $= 65^m$, one leg $= 63^m$; find the other leg.
10. The hypotenuse $= 42.6^m$, one leg $= 19.4^m$; find the other leg.

11. How long must a ladder be to reach the top of a wall 6^m high, if the foot of the ladder is 3.2^m from the bottom of the wall?

12. A ladder 13^m long leans against the top of a wall, its foot being 5^m from the bottom of the wall. How high is the wall?

13. The slant height of a roof $= 14^m$, its vertical height $= 6.2^m$. Find the width of the building.

14. A triangle with the sides 3, 4, 5, is a right triangle. What kind of a triangle is one with the sides 6, 8, 10? with the sides 9, 12, 15? Give other instances.

15. If the perimeter of a right triangle $= 96^m$, and the sides are to each as $3:4:5$, find the sides (see page 50, Exercise 18).

16. If the legs of a right triangle are as $3:4$, and the hypotenuse $= 40^m$, find the legs

17. A rope 41^m long is tied by one end to the top of a mast, and by the other end to a ring on the deck 9^m from the foot of the mast. How high is the mast?

18. The longest side of a meadow in the shape of a right triangle cannot be measured directly, by reason of a swamp. Compute its length, if the other sides are 80.6^m and 160^m.

19. How many meters of wall must be built to fence in a garden in the shape of a right triangle, the legs being 120^m and 35^m?

20. A garden has the shape of a right triangle. The hypotenuse $= 130^m$, one leg $= 87.5^m$. Find (*i.*) the area of the garden; (*ii.*) the distance from the right angle to the hypotenuse.

21. A cannon-ball, fired into the air at an angle of 45° with the horizon, passes over 345^m in one second. How high is it at the end of the second?

22. Two travellers start together from the same place and travel, each 4 miles an hour, one due north, the other due east. When will they be 60 miles apart? How far apart will they be after each has travelled 80 miles?

23. Find the area of an isosceles triangle, if one of the equal sides $= 10^m$ and the base $= 80^m$ (see § **80**).

24. In an isosceles triangle, the base $= 18.4^m$, the perimeter $= 64^m$. Find the area.

25. Prove that in an isosceles right triangle the altitude is equal to half of the base.

26. The area of an isosceles right triangle $= 84^{qm}$; find the base.

27. The altitude of an isosceles right triangle $= 2^m$; find the side of an equivalent square.

28. In an ordinary roof the two sides are 12^m apart at the bottom, and the vertical height is 4.5^m. Find the length of the rafters.

29. The side of an equilateral triangle $= 20^m$; find the altitude and the area (see § **80**).

30. If the altitude of an equilateral triangle $= 15^m$, find one side.

Hint.—The altitude, one side, and half the base, form a right triangle.

31. If $a =$ one side of an eqilateral triangle, prove that the altitude $= \frac{1}{2} a\sqrt{3}$ and the area $= \frac{1}{4} a^2\sqrt{3}$.

32. A garden in the shape of an equilateral triangle contains 600^{qm}; find the side.

33. Find the side of a square equivalent to an equilateral triangle whose side $= 86^m$.

34. A square contains 12^{ha}. Find the side of the equivalent equilateral triangle.

35. The side of a square $= 6^m$. Find its diagonal.

36. What is the side of a square if the diagonal $= 8^m$?

37. The diagonal of a rectangle $= 0.86^m$, one side $= 0.3^m$. Find the other side.

38. The diagonals of a rhombus are 4^m and 6^m. Find its perimeter.

39. How far apart are the opposite corners of a floor 12^m by 18^m?

40. What must be the least thickness of a log from which a rectangular beam 24^{cm} by 40^{cm} can be obtained?

41. How much is the diagonal of a square increased if the side is doubled? trebled? quadrupled?

42. How much is the side of a square increased if the diagonal is doubled? trebled? increased tenfold?

43. Find the less radius of a regular hexagon if the perimeter $= 120^m$.

44. *Compute* the side of a square equivalent to the sum of the squares whose sides are 12^m and 16^m.

45. *Construct* a square equivalent to the sum of two given squares.

46. The side of a square $= 20^{cm}$. Construct a square twice as large. Find the length of its side, (*i.*) by measurement; (*ii.*) by computation.

47. Construct a square equivalent to the sum of the squares whose sides are 30^{cm}, 40^{cm}, 50^{cm}.

48. The side of a square $= 16^{cm}$. Construct a square three times as large.

49. Construct a square equivalent to the difference of two given squares (apply § **74**).

50. Construct a rectangle such that the square upon its diagonal shall be five times the square upon one side.

V.—Transformation of Figures.

§ **144. Definition.**—*To* TRANSFORM *a figure is to change its shape without changing its size; in other words, to change it to an equivalent figure.*

What instance of this kind of change was given in § **129**?

There are two ways of effecting these changes:—

1. By *computation.* Compute such parts of the new figure as are necessary to determine it; we can then construct it, if we wish to do so. Examples of this method occur among the Exercises in Parts III. and IV. of the present chapter (see Part III., Exercises 18, 46, 47, 100, 133, and Part IV., Exercises 27, 33, 34).

2. By direct *construction.* This was the method employed in § **129**, and a few more useful examples will now be added.

§ **145. Problem.**—*To transform a triangle ABC (Fig. 134) into an isosceles triangle.*

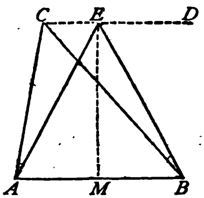


Fig. 134.

Analysis.—The vertices of all triangles equivalent to ABC, and having AB as a base, must lie in the line $CD \parallel AB$ (§ **130**). Of these triangles, that which has its vertex E equidistant from A and B will be isosceles.

Hence give the construction required.

Exercises.—**1.** Construct a triangle, and then transform it into an isosceles triangle.

2. Transform a given triangle into a right triangle.

3. Transform a given triangle into another having the angle 60°.

4. Transform a given parallelogram into a rectangle.

5. Transform a given parallelogram into another having the angle 45°.

§ **146. Problem.**—*To transform a triangle ABC (Fig. 135) into another having a given base b.*

Analysis.—If on AB prolonged we take $AD = b$, and join CD, $\triangle ACD$ will have the given base b, but will be too large by

the $\triangle BCD$. Therefore construct the $\triangle CED \cong \triangle BCD$ (§ **130**, Corollary), and take it from $\triangle ACD$, leaving $\triangle AED$, which is the triangle required.

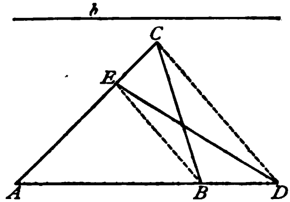


Fig. 135.

Give in full the construction required.

Exercises. — **1.** Solve this problem for the case where the given base b is less than the base of the given triangle.

2. Construct a triangle with the sides 4^{dm}, 6^{dm}, and 8^{dm}, and then transform it, —

(*i.*) Into an isosceles triangle with the base 5^{dm}.
(*ii.*) Into a right triangle with a leg equal to 4^{dm}.
(*iii.*) Into an equilateral triangle.
(*iv.*) Into a triangle with the angle 50° and the base 3^{dm}.

3. Transform a given parallelogram into another having a given side.

§ **147. Problem.** — *To transform a triangle ABC (Fig. 136) into another having a given altitude h.*

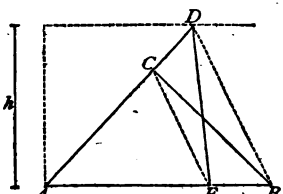


Fig. 136.

Analysis and *construction* to be given by the learner (with the aid of *Fig. 136*).

Exercises. — **1.** Solve this problem for a case in which the new altitude h is less than that of the given triangle.

2. Construct a triangle with the sides 4^{dm}, 6^{dm}, and 8^{dm}, and transform it, —

(*i.*) Into a triangle, the altitude 5^{dm}.
(*ii.*) Into a triangle with the angle 70° and the altitude 4^{dm}.

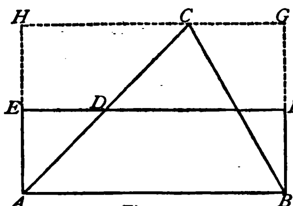


Fig. 137.

§ **148. Problem.** — *To transform a triangle ABC (Fig. 137) into a rectangle.*

Construction.—At A and B erect perpendiculars to AB, and

through D, the middle of AC, draw $EF \parallel AB$. Rectangle $ABEF = \triangle ABC$.

Proof. — Show that each is half of the rectangle $ABHG$.

Exercises. — **1.** Transform a triangle into a parallelogram with the angle 60°.

2. Construct a rectangle with the sides 6^{dm} and 8^{dm}, and then transform it into (*i.*) a triangle; (*ii.*) a right triangle; (*iii.*) an isosceles triangle; (*iv.*) an equilateral triangle.

§ **149. Problem.** — *To transform a rectangle $ABCD$ (Fig. 138) into a square.*

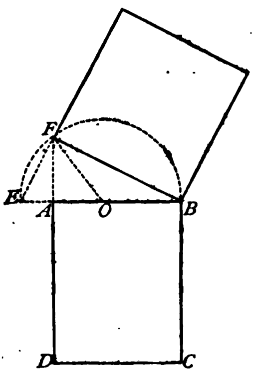


Fig. 138.

Construction. — Make $BE = BC$, and about O, the middle point of BE, describe an arc cutting DA prolonged in F. Join FB, and the square constructed upon FB will be equal in area to the rectangle $ABCD$.

Proof. — EBF is a right triangle. For the triangles OEF and OBF are isosceles (why?); hence $EFO = FEO$, and $OFB = OBF$ (§ **80**). By addition, $EFO + OFB = EFB = FEO + OBF$. But $EFB + FEO + OBF = 180°$ (§ **64**); therefore $EFB = 90°$.

Since EBF is a right triangle, it follows from § **142**, Corollary 3, that $\overline{FB}^2 =$ the area of the rectangle $ABCD$.

Exercises. — **1.** Construct a rectangle with the sides 9^{dm} and 4^{dm}, and transform it into a square. Measure the side of the square. What is its length? What ought its length to be?

2. Construct a triangle with the sides 4^{dm}, 6^{dm}, and 8^{dm}, and transform it into a square.

3. Transform a given parallelogram into a square.

§ **150. Problem.** — *To transform any polygon $ABCDE$ (Fig. 139) into a triangle.*

Construction.—Draw a diagonal AD. Through E draw a line parallel to AD and meeting AB prolonged in F. Join DF. Then is the quadrilateral $BCDF$ equivalent to the pentagon $ABCDE$.

Repeat this construction by drawing the diagonal BD, a parallel through C, and then joining DG; this reduces the quadrilateral $BCDE$ to the equivalent triangle FGD.

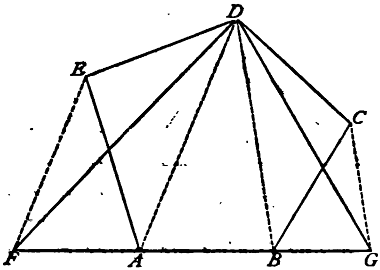


Fig. 139.

Proof.—Left for the learner, with the aid of *Fig. 139* and the analysis of § **146**.

Corollary.—Hence, *any polygon may be transformed into a square.* By what three steps?

This supplies another way of finding the area of a polygonal field. Construct (to scale) the polygon on paper, transform it into a square, and multiply one side of this square by itself.

Exercises.—**1.** Transform a hexagon into a triangle.

2. Transform a pentagon into a square.

3. Construct three equal octagons. Find the area of the first and second by two of the methods given in § **134**, and the area of the third by transformation into a square.

VI.—Partition of Figures.

§ **151.** In buying and selling land it often becomes necessary to run lines of division through estates, forests, fields, etc., dividing them into smaller parts in some assigned way.

This is done either (*i.*) by *computation* (arithmetical partition) or (*ii.*) by *construction* (geometrical partition).

The first method, when practicable, is generally adopted. Let us examine some cases.

§ 152. TRIANGLES. **Case 1.** — A line CD (*Fig. 140*) joining the vertex of a triangle ABC to the middle point of the base AB divides the triangle into two equal parts (**§ 130**, Corollary).

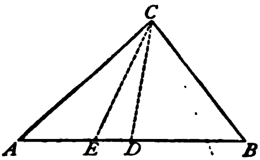


Fig. 140.

Again, if $AE = \frac{1}{3} AB$, and $EB = \frac{2}{3} AB$, then $\triangle ACE = \frac{1}{3} \triangle ABC$, and $\triangle ECB = \frac{2}{3} \triangle ABC$ (why?). And, $\triangle ACE : \triangle ECB = 1 : 2$. In general, —

A line joining the vertex of a triangle to the base divides both the triangle and its base into parts which have the same ratio.

What ratio have the triangles AEC and ABC? the triangles ECB and ABC?

Case 2. — Line of division DE (*Fig. 141*) passes through two sides of the triangle.

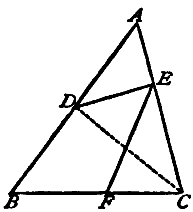


Fig. 141.

Let $AD = \frac{1}{2} AB$, and $AE = \frac{1}{3} AC$; then $\triangle ADC = \frac{1}{2} \triangle ABC$ (Case 1), and $\triangle ADE = \frac{1}{3} \triangle ADC$ (Case 1); therefore $\triangle ADE = \frac{1}{2} \times \frac{1}{3} \triangle ABC = \frac{1}{6} \triangle ABC$.

Again, if $CE = \frac{2}{3} CA$, and $CF = \frac{3}{7} CB$, then (by the same reasoning) $\triangle CEF = \frac{2}{3} \times \frac{3}{7} \triangle ABC = \frac{2}{7} \triangle ABC$; that is, $\triangle CEF : \triangle ABC = 2 : 7$. In general, —

The ratio of the triangle cut off to the entire triangle is equal to the product of the ratios between the parts of the sides that meet at the common vertex.

Case 3. — Hence, if a definite part — say, one-fourth — of a triangle is to be cut off by a line from side to side, this can be done in various ways, because the fraction $\frac{1}{4}$ may be obtained as the product of various pairs of factors; for example, $\frac{1}{2}$ and $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{8}$, $\frac{2}{5}$ and $\frac{5}{8}$, $\frac{5}{6}$ and $\frac{3}{10}$, etc.

If, however, the line of division has to pass through a given point in one side, then its position is determined.

For example, if AD (*Fig. 142*) $= \frac{2}{3} AB$, and the $\triangle ADE$ is to be made equal to $\frac{1}{4} \triangle ABC$, then the line of division will pass through a point in the side AC, since $\triangle ADC$ would be too large (how large is it?); and, since $\frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$, the line will pass through a point E such that $AE = \frac{3}{8} AC$.

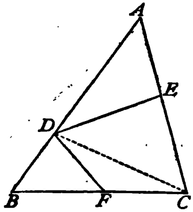


Fig. 142.

How must a line DF be drawn if $\triangle BDF$ is to be made equal to $\frac{1}{6} \triangle ABC$?

§ **153**. Parallelograms. **Case 1.** — Line of division joins two opposite corners. See § **102**.

Case 2. Line of division AE (*Fig. 143*) joins a corner to a point in one side.

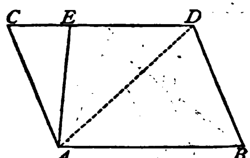


Fig. 143.

Let (*Fig. 143*) $CE = \frac{1}{3} CD$, then $\triangle ACE = \frac{1}{3} \triangle ACD$ (§ **152**, Case 1). But $\triangle ACD = \frac{1}{2}$ the parallelogram $ABCD$ (§ **102**); therefore $\triangle ACE = \frac{1}{6}$ the parallelogram $ABCD$.

CE must be what part of CD, if $\triangle ACE = \frac{2}{9}$ the parallelogram?

Case 3. Line of division joins two opposite sides and is parallel to the other sides. This case is left for the learner to investigate.

How can a rectangle be divided into six equal parts?

Case 4. Line of division EF (*Fig. 144*) joins two opposite sides, and is not parallel to the other sides.

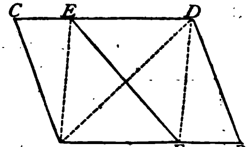


Fig. 144.

Let (*Fig. 144*) $CE = \frac{1}{3} CD$, $AF = \frac{2}{3} AB$, then $\triangle ACE = \frac{1}{3} \triangle ACD$, and $\triangle AEF = \triangle ADF$ (why?) $= \frac{2}{3} \triangle ABD$; therefore $\triangle ACE + \triangle AEF =$ the trapezoid $ACEF = \frac{1}{3} \triangle ACD + \frac{2}{3} \triangle ABD$ $= \frac{1}{3}$ of half the parallelogram $+ \frac{2}{3}$ of half the parallelogram $= \frac{1}{2}$ the parallelogram.

Case 5. Line of division FE (*Fig. 145*) joins two adjacent sides. Let $AE = \frac{1}{2} AB$, $AF = \frac{2}{5} AC$, then (§ **152**, Case 2) $\triangle AEF = \frac{1}{2} \times \frac{2}{5} \times \triangle ABC = \frac{1}{5} \triangle ABC = \frac{1}{10}$ the whole parallelogram.

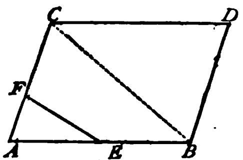


Fig. 145.

§ **154.** We will now give two or three cases in which the partition is effected by direct construction. The proofs of the constructions are left to the ingenuity of the learner.

Problem I.—*To divide a triangle ABC (Fig. 146) from a point D in one side into two equal parts.*

Construction.—Join CD, bisect AB in E, draw $EF \parallel CD$, join DF. DF is the line of division required.

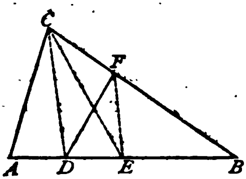


Fig. 146.

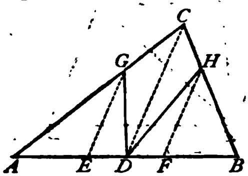


Fig. 147.

Problem II.—*To divide a triangle ABC (Fig. 147) from a point D in one side into three equal parts.*

Construction.—Join CD, trisect AB in E and F, draw EG and FH both $\parallel CD$, join DG and DH; these last are the lines of division required.

Problem III.—*To divide a trapezoid $ABCD$ (Fig. 148) from a point E in one of the parallel sides into two equal parts.*

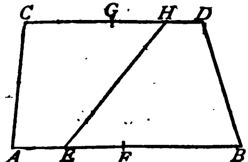


Fig. 148.

Construction.—Bisect the parallel sides in F and G, make $GH = EF$ and join EH: it is the line of division required.

Will this construction divide a parallelogram into two equal parts? See § **153**, Case 4.

How is this problem to be solved, if the line of division is to start from a given point in one of the parallel sides?

§ 155. Practical Exercises.

1. Divide a triangle into five equal parts.

2. Divide a triangle into two parts having the ratio $4:7$.

3. Divide a triangle into three parts which shall be to each other as the numbers 1, 3, 5.

4. Through one corner of a triangle draw a line cutting off a part equal to $\frac{3}{7}$ the entire triangle.

5. How can a triangle containing 120^{qm} be divided into parts containing 30^{qm}, 40^{qm}, and 70^{qm}?

6. From a triangular field worth $1200 cut off a piece worth $400.

7. Through a triangular meadow, a ditch runs from one corner to a point in the opposite side, 264^{yds} from one end, 627^{yds} from the other. Find the ratio of the two parts of the meadow to each other.

8. A piece of woodland in the shape of a triangle cost $6000, but is now worth 50 per cent more than at first. It is required to run a line from one corner to the opposite side, which is 1200 feet long, cutting off a portion worth $1800.

9. A field has the shape of an isosceles right triangle, the equal sides being each 400^{m} long. Through the vertex of the right angle draw a line cutting off a part which shall contain 1 hectar.

Through two sides of a triangle draw a line cutting off, —

10. $\frac{1}{6}$ of the triangle. **11.** $\frac{2}{5}$ of the triangle.

12. $\frac{3}{7}$ of the triangle. **13.** $\frac{4}{9}$ of the triangle.

What ratio does the portion cut off bear to the whole triangle, if the line of division cuts, —

14. $\frac{1}{2}$ from one side and $\frac{1}{4}$ from the other?

15. $\frac{2}{3}$ from one side and $\frac{3}{4}$ from the other?

16. $\frac{5}{6}$ from one side and $\frac{3}{10}$ from the other?

17. The legs of a right triangle are 24^{cm} and 40^{cm}. Find the area of the part cut off by a line which passes through points in the legs distant, respectively, 6^{cm} and 10^{cm} from the vertex of the right angle. What is the ratio of its area to that of the entire triangle?

18. Through the middle point in one side of a triangle draw a line dividing it into parts which shall have the ratio 3 : 5.

19. Through the middle point of one side of a triangle draw lines dividing the triangle into parts which shall be to each other as 1 : 2 : 3.

20. A, B, and C buy a triangular field ABC (*Fig. 149*). A pays \$200; B, \$500; C, \$300. They wish to divide it among themselves, by lines starting at a point D, where there is a valuable spring; A to take the upper part, B the middle, and C the lower. Show how the lines of division should be run if $AD = 180^m$, $DC = 300^m$.

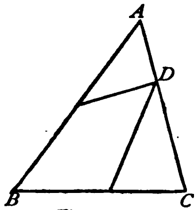


Fig. 149.

21. Divide two sides of a triangle, each into five equal parts, and join in order the points of division. Then find the ratio of each of the smaller triangles thus formed to the entire triangle.

22. What part of a parallelogram is the triangle cut off by a line drawn from one corner to the middle of a side?

23. Cut off $\frac{2}{5}$ of a parallelogram by a line drawn through one corner.

24. From a parallelogram containing 600^{qm} cut a triangle containing 250^{qm}.

25. Divide a rectangle whose sides are 36^m and 20^m, by lines drawn from one corner, into three parts which shall be to one another as 2 : 3 : 4.

26. Divide a parallelogram into seven equal parts.

27. Divide a parallelogram into two parts having the ratio 3 : 5.

28. What part of a parallelogram lies between $\frac{1}{2}$ of one side and $\frac{2}{3}$ of the opposite side?

29. A rectangular field is 500^m by 300^m. A tree is in the longer side, 150^m from one corner. Draw a line, beginning at the tree, which shall cut from the field 1 hectar, adjoining the 150^m part of the longer side.

30. A line joining two adjacent sides of a parallelogram cuts off a triangle whose sides are $\frac{1}{4}$ and $\frac{2}{3}$ of the corresponding sides of the parallelogram. What part of the latter is the triangle?

31. Divide a trapezoid into five equal parts.

32. In a trapezoid $ABCD$ the parallel sides are $AB = 25^m$, $CD = 35^m$; their distance apart $= 20^m$. Divide the trapezoid into two equal parts by a line beginning at a point E in AB, such that $AE = 10^m$.

33. In a quadrilateral $ABCD$, the diagonal $BC = 40^m$, the altitudes of the triangles ABC and BCD are 20^m and 30^m. Divide the quadrilateral by a line through B into two equal parts.

REVIEW OF CHAPTER VII.

SYNOPSIS.

1. Surfaces are measured by choosing a unit and finding how often this unit is contained in the surface to be measured.

2. The *area* of a surface is the number of times the unit is contained in the surface followed by the name of the unit.

3. Units of area (except the acre) are squares whose sides are linear units.

4. The ratio between two units of area is the square of the ratio between the corresponding linear units.

5. Geometry enables us to measure a surface *indirectly* by showing that its area depends upon the length of certain lines.

6. Area of a square = square of one side.

7. Area of a rectangle = its base × its altitude.

8. A parallelogram = a rectangle of the same base and altitude; its area = its base × its altitude.

9. Parallelograms with equal bases and equal altitudes are equivalent.

10. Area of a rhombus = $\frac{1}{2}$ product of its diagonals.

11. A triangle = $\frac{1}{2}$ a parallelogram of same base and altitude; its area = its base × half its altitude.

12. Triangles with equal bases and equal altitudes are equivalent.

13. Area of a trapezoid = $\frac{1}{2}$ sum of parallel sides × altitude.

14. Area of a regular polygon = $\frac{1}{2}$ perimeter × less radius.

15. Area of any polygon may be found by dividing it into triangles (two ways) or into triangles and trapezoids.

16. In every right triangle the square of the hypotenuse is equal to the sum of the squares of the two legs. (Two proofs.)

17. The perpendicular let fall from the vertex of the right angle upon the hypotenuse divides the square of the hypotenuse into two rectangles, equal respectively to the squares of the adjacent legs.

18. A figure may be transformed into another figure, either (*i.*) by *computing* such parts of the new figure as will enable us to construct it, or (*ii.*) by direct *construction*.

19. A square may be found equivalent to any polygon by first changing the polygon to an equivalent triangle, then the triangle to an equivalent rectangle, and then the rectangle to an equivalent square.

20. The division of figures into parts is effected, either (*i.*) by *arithmetical partition*, or (*ii.*) by *geometrical construction*.

21. A line through one corner of a triangle divides the triangle and the opposite side into parts having the same ratio.

22. If a line is drawn between two sides of a triangle cutting off fractional parts of the sides, the product of these fractions give the fractional part which the triangle cut off is of the whole triangle.

23. Nos. **21** and **22** enable us to divide parallelograms and polygons in various ways.

EXERCISES.

1. Compute the areas in square meters (as given below, in the last column to the right) of the following regular polygons, one side of the polygon in each case being supposed to be 1 meter: —

Polygon of	One Side.	Less Radius.	Area.
3 sides	1	0.2887	0.4330
4 sides	1	0.5000	1.0000
5 sides	1	0.6882	1.7205
6 sides	1	0.8660	2.5980
8 sides	1	1.2071	4.8284
10 sides	1	1.5388	7.6942
12 sides	1	1.8660	11.1961

2. Same exercise, one side of each polygon being taken as 2 meters.
Hint. — For the first case (3 sides), the less radius $= 2 \times 0.2887$, and the area $= \frac{3 \times 2 \times 2 \times 0.2887}{2} = 1.732^{qm} = 4$ times the area (0.433) found above for the same polygon when the side is 1^m.

3. Same exercise, one side of each polygon being taken as 3 meters.
Hint. — In the first case (3 sides), the area $= \frac{3 \times 3 \times 3 \times 0.2887}{2} = 9$ times the area (0.433) of the same polygon if the side is 1^m.

4. From the above exercises derive a general rule for finding the area of a regular polygon whose side contains any number (a) of linear units.

5. Two fields of equal value per unit of area have the shape, one of a square, the other of a regular hexagon. If a side of each field $= 136^m$, and the first field is worth $600.00, what is the second field worth? What is the value per square meter? per hectar?

6. Government lands are usually divided into tracts or *townships* 6 miles square; these are subdivided into 36 *sections*, and each section into *half-sections*, *quarter-sections*, *half quarter-sections*, and *quarter quarter-sec-*

tions. Find the area in acres of a township and of each of its subdivisions.

7. A quarter-section is fenced with a six-railed fence, the posts being 12^{ft} apart. The rails cost $40.00 per thousand, the posts $60.00 per thousand. Find the cost of the fence.

8. Of two kinds of paper, equal in quality, the first is 42^{cm} by 30^{cm}, and costs $0.20 a quire; the second is 60^{cm} by 40^{cm}, and costs $0.28 a quire. Which kind is the cheaper?

9. A man buys a rectangular garden 140^{m} by 80^{m}. He surrounds the garden by a wall 0.8^{m} thick, and within the wall by a walk 1.8^{m} wide; and through the middle of the garden he makes paths parallel to the sides, each 1.5^{m} wide. How much land remains for other uses?

10. A piece of land is 160^{m} square. Another piece of the same size has the shape of a rectangle four times as long as it is wide. Find the difference between their perimeters.

11. The perimeter of a right triangle $= 60^{m}$, the hypotenuse $= 25^{m}$. Find the legs and the area.

12. The perimeter of an isosceles triangle $= 24^{m}$, the base $= 6^{m}$. Find the area.

In an isosceles right triangle, —

13. The hypotenuse $= 8^{m}$; find the area.

14. One leg $= 32^{m}$; find the area.

15. The area $= 300^{qm}$; find the sides.

In an equilateral triangle, —

16. One side $= 12^{m}$; find the area.

17. The altitude $= 6^{m}$; find the area.

18. The area $= 144^{qm}$; find the side and the altitude.

19. One side of a square $= 14^{m}$; find the diagonal.

20. The diagonal of a square $= 16^{m}$; find one side.

21. In a rhombus, a diagonal $=$ one side $= 5^{m}$; find the area of the rhombus.

22. Find the angles of the rhombus in the last exercise (see page 110, Exercise 5).

23. A square and a rhombus have the same side, 4^{m}. How much smaller is the latter if its acute angle $= 60^{\circ}$? if its acute angle $= 30^{\circ}$?

24. A rectangular field 200^{m} long contains 1^{ha}. Find the distance apart of two opposite corners.

25. The edge of a cube $= 4^{m}$. Find the distance from a corner in the upper face to the diagonally-opposite corner in the lower face.

26. What method of making a right angle with a cord is suggested by *Fig. 131?*

Construct the following figures: —

27. A triangle, if the area $= 360^{qcm}$, the altitude $= 42^{cm}$.
28. An equilateral triangle, if the area $= 400^{qcm}$.
29. A square, if the diagonal $= 1.2^{m}$.
30. A rectangle, if the area $= 300^{qcm}$, and one side $= 24^{cm}$.
31. A rhombus, if the area $= 240^{qcm}$, and one diagonal $= 30^{cm}$.

Construct a square, —

32. Three times as large as a given square.
33. Two and one-quarter times as large as a given square.
34. Equal to half the difference of two given squares.
35. Equivalent to a given equilateral triangle.
36. Equivalent to a given regular octagon.

Given a square with the side $= 50^{cm}$; compute the side of —

37. An equivalent equilateral triangle.
38. An equivalent regular pentagon.
39. An equivalent regular hexagon.
40. An equivalent regular decagon.

Transform a given triangle, —

41. Into a right triangle.
42. Into an isosceles triangle.
43. Into an isosceles right triangle.
44. Into an equilateral triangle.
45. Into a rectangle with a given side.
46. Into a square.

Transform a square, —

47. Into a parallelogram with a given angle.
48. Into a rectangle with a given side.
49. Into a rhombus with a given side.
50. Transform a given hexagon into a square.
51. What part of a triangle is cut off by a line which bisects two of the sides?
52. Divide a triangle into three equal parts by lines meeting at a given point within the triangle, one line to pass through one corner.
53. Find the point within the triangle from which lines drawn to the corners will divide the triangle into three equal parts.
54. Divide a parallelogram by lines through a corner into three equal parts.
55. Divide a parallelogram by lines through a point in one side into three equal parts.
56. Divide a regular hexagon by lines through one corner into six equal parts.
57. Divide an irregular hexagon by a line through one corner into two equal parts.

CHAPTER VIII.

SIMILAR FIGURES.

CONTENTS.—I. Proportional Lines and Figures (§§ 156–158). II. Similar Triangles (§§ 159–168). III. Problems and Applications (§§ 169–176). IV. Similar Polygons (§§ 177–179).

I.—Proportional Lines and Figures.

§ **156**. Review §§ **37, 38, 112–114**.

The lines a and b (*Fig. 150*) have the ratio $1:3$; the lines c and d have the ratio $2:6 = 1:3$, or the same ratio as the lines a and b. By placing the sign of equality between the equal ratios $a:b$ and $c:d$, we obtain a *proportion*, $a:b = c:d$, which is read *a is to b as c is to d.*

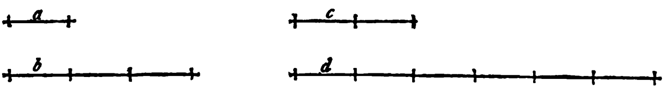


Fig. 150.

Definition.—A PROPORTION *is an equation between two equal ratios.*

The four terms of a proportion need not all be of the same kind; it is sufficient if the two terms of each ratio are alike in kind.

Two areas, for example, may have the same ratio as two lines, and hence form a proportion with them. What instance of this occurs in § **152**?

Again, two angles are to each other as the arcs described from their vertices as centres and included between their sides. In this proportion, the first ratio is that of two angles, the second that of two lines.

Let A, B, C, D be any four magnitudes (that is, things that can be measured), A and B however being alike in kind, and C and D also being alike in kind; and let a and b represent the numerical values of A and B respectively in terms of a common unit, c and d the numerical values of C and D respectively in terms of a common unit; then, if $a:b=c:d$, we say that A, B, C, and D are *in proportion* or are *proportionals*. Of the four *terms* a, b, c, and d which express the proportion, a and d are called the *extremes*, b and c the *means*. If $b=c$, b is called the *mean proportional* between a and d.

§ **157.** A proportion, expressed, as just explained, by the numbers which are the numerical values of the magnitudes forming the proportion, is subject to certain laws, among which the following are useful for our purposes: —

1. If we write the proportion $a:b=c:d$ in the form $\frac{a}{b}=\frac{c}{d}$, and then divide the equation $1=1$ by the equation $\frac{a}{b}=\frac{c}{d}$, we obtain (Axiom V.) $\frac{b}{a}=\frac{d}{c}$, or $b:a=d:c$.

Law I. — *The truth of a proportion is not affected if the terms of each ratio are transposed.*

2. If we multiply both sides of the equation $\frac{a}{b}=\frac{c}{d}$ by $\frac{d}{a}$, we obtain (Axiom IV.), after cancelling common factors, $\frac{d}{b}=\frac{c}{a}$, or $d:b=c:a$. If we multiply $\frac{a}{b}=\frac{c}{d}$ by $\frac{b}{c}$, we obtain $\frac{a}{c}=\frac{b}{d}$, or $a:c=b:d$.

Law II. — *The truth of a proportion is not affected, if the extremes are transposed, or if the means are transposed.*

3. If we multiply both sides of the equation $\frac{a}{b}=\frac{c}{d}$ by the

product $b \times d$, we obtain, after cancelling common factors, $a \times d = b \times c$.

Law III.—*In every proportion the product of the extremes is equal to the product of the means.*

If in a proportion the two means and one extreme are given, how can the other extreme be found?

If in a proportion the two extremes and one mean are given, how can the other mean be found?

Exercises.—**1.** Verify the above laws with the lengths of the lines in *Fig. 150.*

2. Verify the above laws in the case of the proportion $3:9=5:15$.

3. In a proportion, the first, third, and fourth terms are 12, 25, 200; find the second term.

4. Find a fourth proportional to the lengths 24^m, 50^m, and 80^m.

5. Find a length which shall be to 60^m as $5:6$.

6. What relation exists between the side of a square and the sides of the equivalent rectangle?

7. Find the mean proportional between 9^m and 16^m.

8. Show that the mean proportional between a and b is equal to $\sqrt{ab}$.

§ **158.** Let P and Q denote the areas of two squares, two parallelograms, or two triangles; a and b the sides of the squares, or the bases of the parallelograms or the triangles; h and k the altitudes of the parallelograms or of the triangles; then, for—

Two squares, $P=a^2$, $Q=b^2$; hence $\dfrac{P}{Q}=\dfrac{a^2}{b^2}$.

Two parallelograms, $P=a\times h$, $Q=b\times k$; hence $\dfrac{P}{Q}=\dfrac{a\times h}{b\times k}$.

Two triangles, $P=\dfrac{a\times h}{2}$, $Q=\dfrac{b\times k}{2}$; hence $\dfrac{P}{Q}=\dfrac{a\times h}{b\times k}$.

Or, in words,—

Two squares are to each other as the squares of their sides.

State in words the results for parallelograms and for triangles.

In the cases of parallelograms and triangles, if (1) $h = k$, and (2) $a = b$, the preceding equations become, —

$$(1)\quad \frac{P}{Q} = \frac{a}{b}; \qquad\qquad (2)\quad \frac{P}{Q} = \frac{h}{k}.$$

That is, —

(1) *Two parallelograms or two triangles, with equal altitudes, are to each other as their bases.*

(2) *Two parallelograms or two triangles, with equal bases, are to each other as their altitudes.*

Exercises. — 1. The legs of a right triangle are 24^m and 18^m. In another triangle, the base $= 24^m$, the altitude $= 18^m$. Compare the areas of the triangles (that is, find the ratio of the areas).

2. The bases of two triangles of equal altitude are 25^m and 30^m. Compare the areas.

3. If the areas of two triangles of equal altitude are 160^{qm} and 200^{qm}, what is the ratio of their bases? If the base of one $= 32^m$, find that of the other.

4. If the bases of two triangles of equal altitude are 15^m and 19^m, find the base of a triangle with the same altitude and equal in area to their sum.

5. The areas of two triangles with equal bases are 480^{qm} and 300^{qm}. Compare their altitudes. If the altitude of the larger $= 30^m$, find that of the smaller, and also the base of either.

6. Two triangles have equal bases, and the altitude of one is ten times that of the other. Compare their areas.

7. If two triangles have equal areas, compare their altitudes when the base of one triangle is (*i.*) twice, (*ii.*) four times, (*iii.*) one-half, (*iv.*) three-fifths, that of the other.

8. The altitudes of two equivalent triangles are 15° and 20^m. Compare their bases.

9. Solve Exercises 6, 7, and 8, substituting "parallelograms" for "triangles."

10. Compare a triangle with a parallelogram of the same base and altitude.

11. A triangle and a parallelogram have equal areas. Compare (*i.*) their bases if they have equal altitudes; (*ii.*) their altitudes if they have equal bases.

12. Compare the areas of a rhombus and the rectangle whose sides are bisected by the corners of the rhombus (see *Fig. 124*).

13. A square and a rhombus have equal perimeters. Compare their areas if the altitude of the rhombus is two-thirds its side.

14. A square and a rhombus have equal areas. Compare their perimeters if the altitude of the rhombus is one-half its side.

15. Compare the areas of a square and an equilateral triangle having the same side.

16. A square and an equilateral triangle have equal areas. Compare their sides.

17. Compare the areas of a square, a regular pentagon, and a regular hexagon, all having the same side.

II.—Similar Triangles.

§ **159.** Two triangles which have the same shape and differ only in size are called SIMILAR triangles.

The sign of similarity is $\sim$. $\triangle ABC \sim \triangle DEF$ is read, the triangle ABC is similar to the triangle DEF.

In order to discover the chief properties possessed by similar triangles, take on one side AK (*Fig. 151*) of an angle KAL any number, say five, equal parts, AB, BD, DF, FH, HK, and through the points B, D, F, H, K, draw parallel lines cutting AL in the points C, E, G, I, L; then (§ **112**) $AC = CE = EG = GI = IL$.

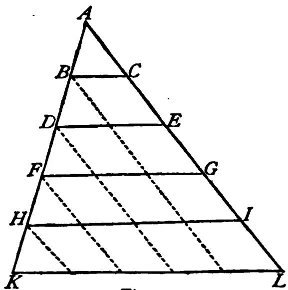


Fig. 151.

The triangles ABC, ADE, etc., differ in size, but have the same shape; hence are similar.

Now compare the angles and sides of any two of these triangles, as ADE and AKL. The angles are equal; for the angle A is common; and of the other angles, $ADE = AKL$, and $AED = ALK$ (§ **56**). As for the sides, it is evident from the construction that $AD : AK = 2 : 5$, and likewise that $AE : AL = 2 : 5$. The same ratio also exists between the sides DE and KL; for, if we draw through B, D, F, and H lines parallel to AL, these lines will divide (by § **112**) DE into two equal parts, and KL into five equal parts,

and the parts of DE will be equal to those of KL (§ **102**, Corollary 1); therefore, $DE : KL = 2 : 5$. Hence, $AD : AK = AE : AL = DE : KL$; that is to say, the sides opposite equal angles in the triangles are proportional.

The sides opposite equal angles are termed *corresponding* or *homologous* sides.

We should obtain like results if the angle A had any other value, or if the parallels BC, DE, etc., had any other direction, and also whatever be the length or number of the equal parts AB, BD, etc.

Therefore, —

Theorem I. — *Two similar triangles are mutually equiangular, and have their homologous sides proportional.*

Corollary. — By transposing (§ **157**, Law II.) the means of the proportion $AD : AK = AE : AL$, we obtain $AD : AE = AK : AL$. This shows that, *in similar triangles the sides which contain equal angles have the same ratio.*

Two other theorems also follow from the preceding investigation: —

Theorem II. — *If a line is drawn through two sides of a triangle, parallel to the third side, a smaller triangle is formed which is similar to the given triangle.*

Theorem III. — *A line drawn through two sides of a triangle, parallel to the third side, divides those two sides proportionally.*

Exercises. — **1.** The converse of Theorem III. is also true. State it, and illustrate it by means of numbers.

2. The three sides of a triangle are 84^m, 120^m, and 100^m. Through a point in the first side, 28^m from the intersection of the first and second sides, a line is drawn parallel to the third side. Find where it will cut the second side. Find, also, the ratio between the triangle cut off and the entire triangle (§ **152**).

3. If the sides of a triangle are 16^m, 24^m, and 32^m, and a line is drawn from the first side to the second side, connecting points distant 5^m and 7^m, respectively, from the intersection of the first and second sides, will this line be parallel to the third side?

4. Draw two similar triangles. Also, two *equal* triangles. Can two triangles be equal without being similar? Can they be similar without being equal?

§ **160.** It appears from the last section that the similarity of two triangles involves six conditions: the equality of three pairs of angles and the equality of the ratios between three pairs of sides. What six conditions does the equality of two triangles involve?

We have seen (§§ **72–75**) that, in general, from the equality of three parts in two triangles, we can conclude that the triangles are equal in all respects. We shall now proceed to show that if two triangles fulfil certain conditions out of the six above mentioned as belonging to similar triangles, then these triangles *must* fulfil the remaining conditions, and hence must be similar to each other.

§ **161.** In the triangles ABC and DEF (*Fig. 152*) let the angle $A = D$, $B = E$, and therefore $C = F$. Make $CG = FD$, and draw $GH \parallel AB$. Then $\triangle GHC \cong \triangle DEF$ (I. Law of Equality). But $\triangle ABC \sim \triangle GHC$ (§ **159**, Theorem II.). Therefore $\triangle ABC \sim \triangle DEF$.

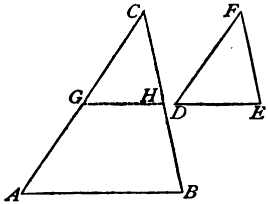


Fig. 152.

Hence,—

Theorem (I. Law of Similarity).—*Two triangles are similar if they are mutually equiangular.*

Exercises.—**1.** Show that two triangles are similar if two angles of one are equal respectively to two angles of the other.

2. What condition suffices to make two isosceles triangles similar? two right triangles?

3. Why are two equilateral triangles always similar?

4. Construct three similar triangles, each having the angles 60° and 80°.

5. Construct upon a given line AB a triangle similar to a given triangle CDE.

6. Two triangles are similar if their sides, taken two by two, are parallel.

7. Two triangles are similar if their sides, taken two by two, are perpendicular to each other. Point out the homologous sides.

Hints.—Revolve one of the triangles about a corner through 90°; then its sides become parallel respectively to the sides of the other triangle. See Exercise 6.

§ **162.** In the triangles ABC and DEF (*Fig. 152*), let $AC:DF = BC:EF$, and let also the angle $C = F$. Make $CG = DF$, and draw $GH \parallel AB$; then $AC:GC = BC:HC$ (§ **159**, Theorem III.) Since the first three terms of this proportion are the same as the first three of the proportion assumed at the start, the fourth terms must also be equal; therefore $CH = EF$. Then $\triangle GHC \cong \triangle DEF$ (II. Law of Equality). But $\triangle ABC \sim \triangle GHC$ (why?): therefore $\triangle ABC \sim \triangle DEF$; and hence, —

Theorem (II. Law of Similarity). — *Two triangles are similar if they have an angle equal, and the sides which include the angle proportional.*

§ **163.** In the triangles ABC and DEF (*Fig. 152*), let $AC:DF = BC:EF$, $AC > BC$, $DF > EF$, and $B = E$. Make $CG = DF$, and draw $GH \parallel AB$; then $AC:GC = BC:HC$. This proportion and the proportion assumed at first have the first three terms equal, therefore their fourth terms must be equal, or $CH = EF$. Whence, $\triangle GHC \cong DEF$ (III. Law of Equality). But $\triangle ABC \sim \triangle GHC$, therefore $\triangle ABC \sim \triangle DEF$. Hence, —

Theorem (III. Law of Similarity). — *Two triangles are similar if two sides of one are proportional to two sides of the other, and the angles opposite the greater sides are equal.*

§ **164.** In the triangles ABC and DEF (*Fig. 152*), let

$$AC:DF = BC:EF,$$

and

$$AC:DF = AB:DE.$$

Make $CG = DF$, and draw $GH \parallel AB$, then —

$$AC:GC = BC:HC;$$

$$AC:GC = AB:GH.$$

In the first and third of these proportions, the first three terms are equal, therefore the fourth terms must also be equal; that is, $CH = EF$. For the same reason it follows from the second and fourth proportions that $GH = DE$. Hence, $\triangle GHC \cong \triangle DEF$

(IV. Law of Equality); but $\triangle ABC \sim \triangle GHC$, therefore $\triangle ABC \sim DEF$. That is,—

Theorem (IV. Law of Similarity).—*Two triangles are similar if their sides, taken in order, are proportionals.*

Corollary.—If each side of a triangle is two, three, four, etc., times the homologous side of a similar triangle, then the sum of the sides, that is, the perimeter, of one triangle is two, three, four, etc., times the perimeter of the other triangle; that is,—

In similar triangles the perimeters are to each other as any two homologous sides.

Exercise.—Verify the truth of this Corollary by two triangles, the sides of one being 2^m, 4^m, 5^m, and those of the other 6^m, 12^m, 15^m.

§ **165.** Let the parallels AC, DF (*Fig. 153*), be cut by lines OA, OB, OC, drawn through a common point O.

$$\triangle OAB \sim \triangle ODE,$$

and

$$\triangle OBC \sim \triangle OEF$$

(§ **56**, and I. Law of Similarity). Therefore,—

$$\frac{OB}{OE} = \frac{AB}{DE} \text{ and } \frac{OB}{OE} = \frac{BC}{EF}.$$

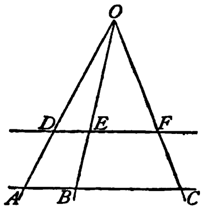


Fig. 153.

Whence (Axiom I.), $\frac{AB}{DE} = \frac{BC}{EF}$, or $\frac{AB}{BC} = \frac{DE}{EF}$.

The same reasoning might be applied to any number of parallels and lines intersecting them. Therefore,—

Theorem.—*Lines drawn through a common point divide parallels into proportional parts.*

Exercises.—**1.** If one of the parallels is divided into equal parts, how will the others be divided?

2. If a parallel to AC (*Fig. 153*) is drawn above O, cutting the lines through O prolonged, will the theorem still hold true? Can you prove?

§ **166.** In the right triangle ABC, right angled at C (*Fig. 154*) draw $CD \perp AB$; it divides AB into two parts or *segments*. The smaller right triangles ACD and BCD are each equiangular with respect to the triangle ABC (why?); therefore they are equiangular with respect to each other. Hence (I. Law of Similarity), —

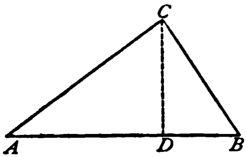


Fig. 154.

$\triangle ABC \sim \triangle ACD$; therefore $AB : AC = AC : AD$.
$\triangle ABC \sim \triangle BCD$; therefore $AB : BC = BC : BD$.
$\triangle ACD \sim \triangle BCD$; therefore $AD : CD = CD : BD$.

Theorems. — *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:* —

I. *Either leg is a mean proportional beeween the hypotenuse and the adjacent segment.*

II. *The perpendicular is a mean proportional between the two segments of the hypotenuse.*

Exercises. — In a right triangle let a and b denote the legs, c the hypotenuse, h the altitude with hypotenuse for base, p and q the segments of the hypotenuse. Find the other quantities, if, —

1. $a = 4.5^m$, $h = 3.6^m$.
2. $h = 0.8^m$, $p = 1.5^m$.
3. $a = 9.6^m$, $p = 0.784^m$.
4. $c = 1250^m$, $p = 336^m$.

§ **167.** Let (*Fig. 155*) $\triangle ABC \sim \triangle DEF$, and draw the altitudes CG and FH corresponding to two homologous sides AB and DE taken as bases.

By hypothesis, $AB : DE = AC : DF$. The right triangles ACG and DFH are similar (why?), therefore $AC : DF = CG : FH$. Hence $AB : DE = CG : FH$. That is, —

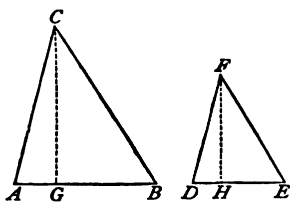


Fig. 155.

Theorem. — *In two similar triangles, any two homologous sides taken as bases are to each other as the corresponding altitudes.*

§ **168.** Let P and Q denote the areas, a and b the bases, h and k the corresponding altitudes of two similar triangles.

Then (§ **158**), $\frac{P}{Q} = \frac{a \times h}{b \times k} = \frac{a}{b} \times \frac{h}{k}.$

But (§ **167**) $\frac{a}{b} = \frac{h}{k}.$

If for the factor $\frac{h}{k}$ in the first equation we substitute its equal $\frac{a}{b}$, the equation becomes,—

$$\frac{P}{Q} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}.$$

Since a and b may be any two homologous sides taken as bases, therefore,—

Theorem.—*Similar triangles are to each other as the squares of their homologous sides.*

Exercises.—**1.** If the homologous sides of one triangle are each one-fifth those of a similar triangle, compare the areas of the triangles.

2. The areas of two similar triangles are 400^{qm} and 1296^{qm}; find the ratio of two homologous sides.

3. Two triangular gardens are similar in shape. What is the easiest way (*i.*) to *compare* their areas? (*ii.*) to *compute* both their areas?

4. A field has the shape of an equilateral triangle. From one corner I wish to cut off a similar triangle equal in area to one-sixteenth of the whole; how must the line of division be run?

Draw a triangle; then construct a similar triangle equal in area to—

5. 4 times that of the first triangle.

6. 2 times that of the first triangle.

7. $\frac{1}{9}$ that of the first triangle.

8. $\frac{9}{16}$ that of the first triangle.

9. Can you prove the theorem of this section by a method suggested by § **152**, Case 2?

Hints.—Let a line cut $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{5}$, or any other fraction, from each of two sides of a triangle; and use § **159**, Exercise 1, and Theorem II.

III.—Problems and Applications.

§ **169. Problem.**—*To find the fourth proportional to three given lines a, b, c (Fig. 156).*

Construction.—Make any angle O; on its sides take $OA = a$, $OB = b$, $OC = c$; join AC, and draw $BD \parallel AC$. OD is the fourth proportional required.

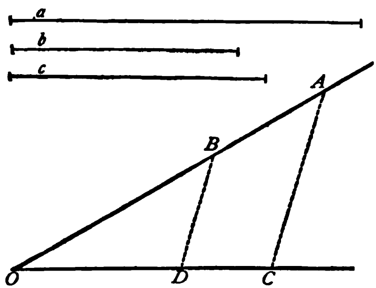


Fig. 156.

Proof.—Apply § **159**, Theorem III.

Exercises.—Give the construction for the following cases:

1. $b < a$, and $c > a$.
2. $b < a$, and $c = a$.
3. $b < a$, and $c = b$.
4. $b > a$, and $c > b$.
5. $b > a$, and $c > a$.
6. $b > a$, and $c = a$.
7. $b > a$, and $c = b$.

§ **170. Problem.**—*To find a mean proportional to two given lines a and b (Fig. 157).*

§ **169** furnishes one solution (see Exercises 3 and 7). The more common solution is as follows:—

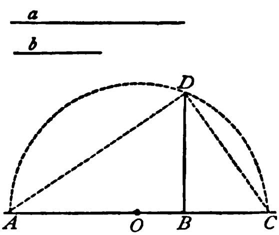


Fig. 157.

Construction.—Draw a straight line, and take $AB = a$, $BC = b$. At B erect a perpendicular. Bisect AC in O, and with a radius equal to OA describe an arc cutting the perpendicular in D. BD is the mean proportional required.

Proof.—Prove, as in § **149**, that $ADC = 90°$; then apply § **166**.

§ **171.** **Problem.** — *To reduce or enlarge given lines a, b, c, etc.* (*Fig. 158*), *in a given ratio.*

Construction. — Let the ratio be 3 : 4; that is, let it be required to reduce the lines each to three fourths its present length.

Draw a straight line and take on it four equal parts, of which OA contains four, OB three. At A and B erect perpendiculars; make $AC = a$, $AD = b$, $AE = c$, etc. Join OC, OD, OE, etc. BF, BG, BH, etc., are the reduced lengths required.

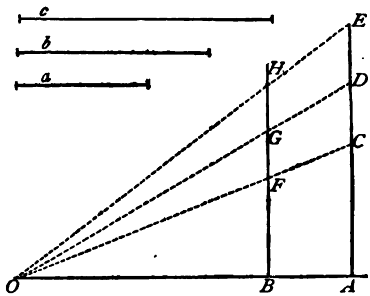


Fig. 158.

Proof. — $\triangle OBF \sim \triangle OAC$, $\triangle OBG \sim \triangle OAD$, etc. (why?). Then apply § **159**, Theorem I.

Exercises. — **1.** Draw four lines and reduce them in the ratio 3 : 5.

2. Draw four lines and enlarge them in the ratio 5 : 2.

§ **172.** **Problem.** — *To divide a given line into equal parts.*

One solution of this problem has been given (§ **113**); but, since that solution requires the construction of several parallels, it is tedious and liable to lead to errors.

The following construction is more simple: —

Construction. — Let it be required to divide MN (*Fig. 159*) into five equal parts. Draw any line MO through M; then through a point P of this line draw a line $PQ \parallel MN$, and take on this parallel any five equal parts ending at a point Q.

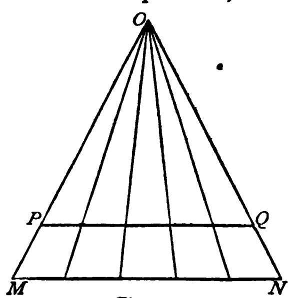


Fig. 159.

Through N and Q draw a line cutting MO at O. Lastly, draw lines through O and the points of division of PQ. These lines will divide MN into five equal parts.

Proof. — Apply § **165**.

Exercises. — 1. Draw a line and divide it into eight equal parts.

2. Divide the line AB (*Fig. 160*) into ten equal parts.

Solution. — Here the parts are so short that if obtained in the way above described they would be indistinct. In this case proceed thus: Erect at A and B perpendiculars, lay off upon each ten equal parts, and through the points of division draw lines parallel to AB. If now we draw the diagonal AD, then the distances from the points marked 1, 2, 3, 4, etc., to the line BD will be equal to $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, etc., of the line AB. The proof is left to the learner.

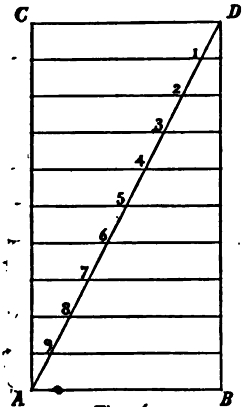


Fig. 160.

3. When (in making maps, plans, etc.) many lines have to be very much reduced in length, it becomes often convenient to employ scales of equal parts, known among surveyors as PLATTING SCALES. Explain how such a scale (shown in *Fig. 161*), containing one thousand parts, divided decimally, is constructed and used.

Solution. — Draw a straight line, and take on it ten equal parts, $AB = BC = CD$, etc.,[1] each of these lengths representing one hundred units of length (meters). Divide AB into ten equal parts, each part representing therefore ten units of length. Lastly, to represent single units of length, draw above AB ten equidistant lines parallel to AB, erect perpendiculars at the points A, B, C,

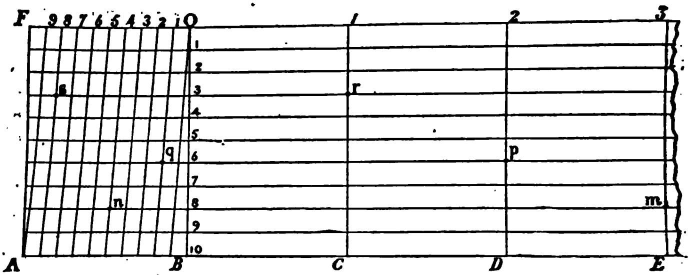


Fig. 161.

[1] Only a part of the scale can be shown in the figure.

etc., and, after having subdivided FO into ten equal parts, join the points of division of FO with those of AB by lines drawn obliquely, as shown in the figure.

The perpendicular OB, divided into ten equal parts by the parallels is numbered from 1 to 10, beginning at O; then it follows (see Exercise 2) that the distance from 1 to the first oblique line measured on the parallel = 1 unit of length, the distance from 1 to the second oblique line = 11 units, etc.; and the distance from 2 to the first oblique line = 2 units, the distance to the second oblique line = 12 units, etc.; and similarly for the other points of division of OB.

The point marked O is the zero of the scale; hundreds are read off to the right, tens to the left, and units on the vertical line.

From this explanation it is easy to see that a length of 348 meters measured on the ground is represented upon the scale by the line mn; also, that the length pq corresponds to a distance of 216 meters, the length rs to 183 meters, etc.

4. Construct, with the aid of the scale in *Fig. 161*, a triangle whose sides are 137^m, 160^m, and 225^m.

5. Draw a triangle, and determine the lengths of its sides by means of the scale in *Fig. 161*.

6. Construct a scale of one hundred parts, divided decimally.

§ **173.** The measurement of the distance of an inaccessible object is one of the most interesting applications of the laws of Geometry. It gives us a good idea of the nature of those methods which have enabled astronomers to measure the distance from the earth to the moon, to the sun, and even to some of the fixed stars.

We have already seen (§§ **95–98**) how distances may be measured indirectly by means of *equal* triangles. Now it is clear that, if equal triangles are employed, the greater the distance to be measured the larger the triangle which must be constructed; so that, when the distance is considerable, the construction of the triangle becomes very inconvenient, and in many cases (for instance, the distance from the earth to the moon) utterly impossible. In such cases other methods must be found; and these methods are supplied by the laws of *similar* triangles.

In fact, the second method given for solving the problems of § **95** and § **98** is based upon the properties of similar triangles. In § **95**, Method II., we construct on paper to a reduced scale a *fac-simile* of the actual lines on the ground. What is this *fac-simile* but a small triangle similar to the much larger one which contains the distance sought? In other words, the triangles ABC and abc

(*Fig. 88*) are similar; hence $ac : ab = AC : AB$, a proportion which, by making (as on page 105) $ac = 16^{cm}$, $ab = 24^{cm}$, $AC = 800^{m}$, becomes $16 : 24 = 800 : AB$; whence $AB = \frac{24 \times 800}{16} = 1200^{m}$.

Representing lines to scale on paper, all angles remaining unchanged, always gives a figure similar to that formed by the unreduced lines, and is termed PLATTING TO SCALE.

Some examples will now be given of the indirect measurement of distances by means of similar triangles.

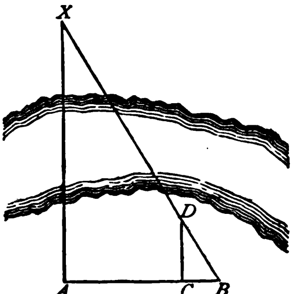


Fig. 162.

§ 174. Problem. — *To measure the distance of two points A, X, separated by a river.*

At the point A (*Fig. 162*), supposed accessible, erect (with a surveyor's cross or other means) a line $\perp AX$, and measure its length to any point B. Suppose, for example, that $AB = 120^{m}$; take $BC = 10^{m}$, and at C erect a second perpendicular CD, meeting the line BX at D. Measure CD; let its length be 18^{m}. Then it follows that, *in all right triangles similar to BCD — that is to say, in all right triangles having an acute angle equal to DBC — the ratio of the legs must be* 18 : 10.[1]

Among these similar right triangles is the triangle ABX; therefore, $AX : AB = 18 : 10$; whence, $AX = 216^{m}$.

Exercises. — 1. In *Fig. 162*, BC is made equal to 10^{m} solely for convenience of computation. If $BC = 15^{m}$, what would DC be found to be?

2. If A is on the edge of the river, can you find AX by this method? What change in the position of the triangle BCD is necessary? Give the construction for this case.

3. How can AX be found by means of two equal right triangles?

4. How can AX be found by means of an isosceles right triangle?

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[1] In Trigonometry, 18 : 10, or $\frac{18}{10}$, is called the TANGENT of the angle DBC.

§ **175. Problem.** — *To find the distance between two inaccessible points, X, Y.*

Case 1 (*Fig. 163*). — Choose an accessible point, A, and measure the distances AX, AY, by the preceding problem. Upon AX measure a length Ax equal to any fractional part (as $\frac{1}{10}$, $\frac{1}{20}$, etc.) of AX, and upon AY measure Ay equal to the *same* fractional part of AY. Lastly, measure xy, which is parallel to XY (why?). Then the distance XY can be found from the proportion $Ax : AX = xy : XY$.

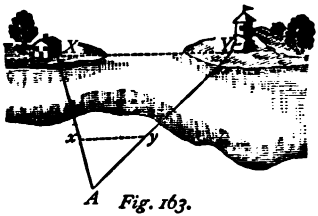

Fig. 163.

If, for example, $Ax = \frac{1}{10} AX$, then $XY = 10\,xy$.

Case 2. — Where the line XY prolonged is accessible (*Fig. 164*). Erect at M, an accessible point, a line $\perp MX$, and mark the points N where this line intersects the perpendicular let fall from Y, and P, where it cuts XY prolonged. Measure MP and NP; let, for example, $MP = 360^m$, $NP = 200^m$. Take $PO = 10^m$; at O erect $OQ \perp PM$, and measure the hypotenuse PQ of the right triangle POQ. Suppose, for example, that $PQ = 24^m$; then, *in all right triangles having an acute angle equal to OPQ the ratio of the leg corresponding to OP to the hypotenuse is* $10 : 24$.[1]

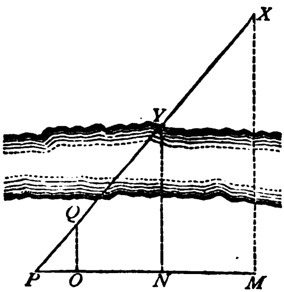

Fig. 164.

PMX and PNY are two such triangles; therefore, —

$$10 : 24 = PM, \text{ or } 360^m : PX, \text{ whence } PX = 864^m;$$
$$10 : 24 = PN, \text{ or } 200^m : PY, \text{ whence } PY = 480^m;$$

and $XY = PX - PY = 384^m$.

[1] In Trigonometry, the ratio $10 : 24$, or $\frac{5}{12}$, is called the COSINE of the angle OPQ.

Exercises. — 1. What is the advantage in Case 2 of making $PO = 10^m$ rather than any other number?

2. Explain how, with the aid of an instrument for measuring angles, the distance XY in *Fig. 163* may be found by platting a similar triangle on paper. By this method how many lines must be measured on the ground? How many by the other method?

3. Given (*Fig. 163*) $AX = 600^m$, $AY = 720^m$, $Ax = \frac{1}{20} AX$, $Ay = \frac{1}{20} AY$, $xy = 21.18^m$, $XAY = 36°$. Find XY (*i.*) by a direct proportion; (*ii.*) by platting a similar triangle to scale on paper. How nearly do the results agree? Why do they not exactly agree?

4. *Fig. 165* illustrates another method of measuring an inaccessible line XY, — a method by which only one line AB has to be measured on the ground. Can you explain the method?

Fig. 165.

§ **176. Problem.** — *To measure the vertical height of an object (a tower, tree, church-spire, etc.)*

1st Method. — By means of a shadow.

Suppose that we wish to find the height AX of a tree (*Fig. 166*) which casts the shadow AB on the ground. Fix a staff vertically in the ground near the tree, and measure its height mn. Measure, also, the lengths of the shadows no cast by the staff, and AB cast by the tree. Now the ray of light XB is parallel to the ray mo: whence it follows that ABX and mno are similar trian-

gles (why?). In other words, *the shadows cast at the same time by different objects are proportional to the heights of the objects.*

Therefore $no : AB = mn : AX$.

If $no = 0.9^{m}$, $AB = 24.9^{m}$, $mn = 1.17^{m}$, then, —

$$0.9 : 24.9 = 1.17 : AX,$$

whence $AX = 32.37^{m}$.

NOTE.—We may also reason thus: if 90^{cm} of shadow correspond to 117^{cm} of height, then 1^{cm} of shadow corresponds to $\frac{117}{90} = 1.3^{cm}$ of height; therefore 2490^{cm} of shadow will correspond to $1.3^{cm} \times 2490 = 3237^{cm} = 32.37^{m}$.

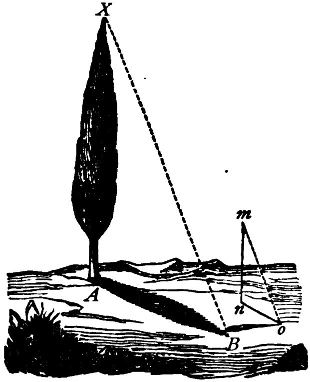

Fig. 166.

2d Method.—With the aid of an instrument for measuring angles.

Let, for example, the height of the tower (*Fig. 167*) be required. Place the instrument at a point B, some distance from the foot of the tower, and measure the angle abX, or *angle of elevation*, as it is termed. Measure AB, the distance from the instrument to the foot of the tower. Then construct on paper a right triangle similar to abX, and the side of this triangle homologous to aX will give *to the scale employed* the height aX, to which we must add Aa, in order to obtain the entire height.

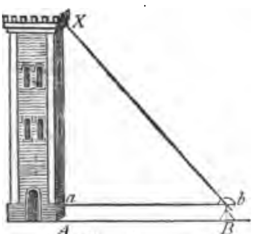

Fig. 167.

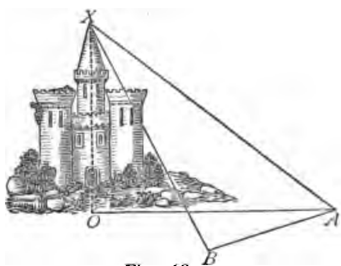

Fig. 168.

3d Method.—When the base of the object is inaccessible, choose

a convenient place near the object, and measure a *base* line AB (*Fig. 168*). Then measure the angles at the base, B and BAX, and also the angle of elevation of the object XAO, and construct on paper a triangle similar to XBA. This will give *to scale* the length AX. Then, knowing AX and the acute angle XAO, construct on paper the right triangle OAX, which will give to scale the height OX required.

Here the unknown height OX depends on the construction of two similar triangles, the first similar to the triangle ABX, the second to the right triangle OAX. If the scale of reduction in both cases is 1^{mm} to 1^{m}, the number of millimeters contained in the line on paper which represents OX will be the same as the number of meters in OX.

Exercises. — 1. A monument casts a shadow of 44^{m} at the same time that a rod 2^{m} high casts a shadow of 1.33^{m}; find the height of the monument.

2. In the 2d Method, why is the construction of a triangle on paper avoided by placing the instrument so that the angle of elevation shall be equal to 45°?

3. What instruments are required in the 3d Method?

4. Find the height of the building (*Fig. 168*) if $AB = 250^{m}$, angle $B = 25^\circ\ 30'$, angle $XAB = 104^\circ$, and angle $XAO = 41^\circ\ 30'$.

IV.—Similar Polygons.

§ **177.** Two polygons are said to be SIMILAR if they have the same shape.

In order to see what are the properties of similar polygons, divide the polygon $ABCDE$ (*Fig. 169*) into triangles by drawing the diagonals AC, AD; then begin at any point M of AB, and make a new polygon by drawing $MN \parallel BC$, $NO \parallel CD$, $OP \parallel DE$. The new polygon $AMNOP$ is smaller than the original polygon, but has the same shape, and is therefore similar to it.

By the same process, we can make a polygon $AQRST$ larger than the polygon $ABCDE$, and similar to it.

Now the triangles AMN, ABC, AQR, are similar (why?); so are the triangles AON, ADC, ARS; and lastly, the triangles AOP, ADE, AST. Whence it follows that,—

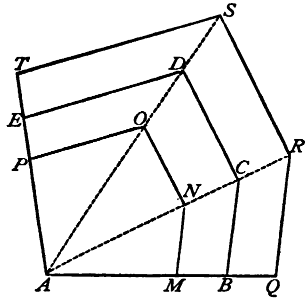

Fig. 169.

Theorem I.—*Similar polygons are composed of the same number of triangles similar to each other and similarly placed.*

From this it follows, that similar polygons are mutually equiangular. Why, for example, is the angle AMN equal to the angle ABC? the angle MNO equal to the angle BCD?

Compare now the sides in any two of the polygons, say the two smaller ones. From the similar triangles AMN and ABC,—

$$AM : AN = MN : BC,$$

and also;
$$AN : AC = MN : BC;$$

and from the similar triangles ANO and ACD,

$$AN : AC = NO : CD.$$

From the second and third proportions, it follows (Axiom I.) that

$$MN : BC = NO : CD.$$

In this way, whatever be the number of sides, it may be shown that the sides similarly placed are always proportional.

Theorem II.—*Similar polygons are mutually equiangular, and have their homologous sides proportional.*

Corollaries.—1. *Two regular polygons of the same number of sides are similar.* Why?

2. If each side of a polygon is 2, 3, 4, etc., times as great as the homologous side of a similar polygon, then the sum of the sides, that is, the perimeter of the first polygon, must be 2, 3, 4, etc., times as great as the sum of the sides, or the perimeter, of the second polygon. That is to say, *the perimeters of similar polygons are to each other as any two homologous sides.*

§ **178.** If two similar polygons, the sides of one of which are double those of the other, are divided into triangles, by drawing diagonals from the vertices of two equal angles, then each triangle in the first polygon is (by § **164**) four times as great as the corresponding triangle on the second polygon; hence the sum of all the triangles in the first polygon — in other words, the area of the first polygon — is four times as great as the sum of all the triangles, or the area, of the second polygon. Hence the areas of the two polygons are to each other as 4 : 1. But this is the ratio of the squares of any two homologous sides. A like result would follow if the sides had any other ratio than 2 : 1. Hence, —

Theorem. — *Similar polygons are to each other as the squares of their homologous sides.*

Therefore, if we have a polygonal figure on the ground, and construct a similar figure on paper with its sides reduced to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{10}$, etc., of their lengths, the area of the figure on the paper will be $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{100}$, etc., the area of the larger and similar figure on the ground.

Exercises. — **1.** Compare the areas of two similar polygons with their perimeters (see § **177**, Corollary 2).

2. Prove that the areas of regular polygons of the same number of sides are to each other, (*i.*) as the squares of their sides; (*ii.*) as the squares of their perimeters.

3. If in two hexagonal parks a side of one is four times a side of the other, how much larger is one park than the other? Find, also, the ratio of their perimeters.

§ **179. Problem.** — *To construct a polygon similar to a given polygon.*

There are various methods of solving this problem.

1st Method. — When one side FG of the required polygon is given.

Solution. — By means of § **177**, Theorem I. Divide the given polygon $ABCDE$ (*Fig. 170*) into triangles by diagonals; then

construct upon the given length FG, homologous to AB, a series of triangles similar to those of the given polygon respectively, and similarly placed. The polygon $FGHKL$ is the polygon required.

Give the construction in full.

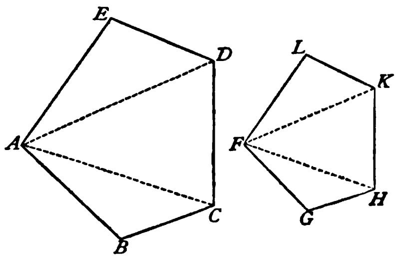

Fig. 170.

2d Method. — One side FG being given, as before.

Solution. — On one side AB of the given polygon $ABCDE$ (*Fig. 171*) take $AM = FG$, construct the polygon $AMNOP \sim ABCDE$ (how is this done?), and then construct (by § **117**) upon the side FG the polygon $FGHKL$ $\cong$ the polygon $AMNOP$.

Give the construction in full.

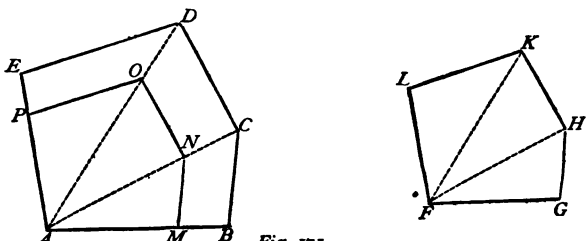

Fig. 171.

3d Method. — When the homologous sides of the two polygons are to be to each other in a given ratio.

Solution. — Choose any point O, either within or without the given polygon (see *Figs. 172* and *173*), and join it to all the corners by straight lines. Divide one of these lines at a point M, so that $OM : OA =$ the given ratio, and then draw $MN \parallel AB$, $NP \parallel BC$, etc. $MNPQR$ is the required polygon.

In *Fig. 172*, $OM : OA = 1 : 3$; in *Fig. 173*. $OM : OA = 3 : 5$. Give the construction and proof for both cases.

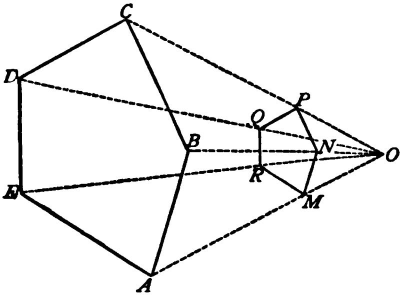

Fig. 172.

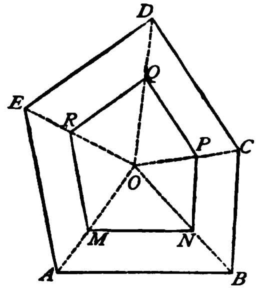

Fig. 173.

Exercises. — **1.** Given a hexagon; construct a similar but larger hexagon.

2. Given an octagon; construct a similar but smaller octagon.

3. Upon the lengths 20^{cm} and 30^{cm} as homologous sides construct two similar pentagons.

4. Draw a polygon, and then construct another whose sides shall be,—

(*i.*) one-half those of the first polygon;

(*ii.*) two-thirds those of the first polygon;

(*iii.*) twice those of the first polygon;

(*iv.*) three and one-half times those of the first polygon.

5. Draw a polygon, and then construct a similar polygon,—

(*i.*) one-ninth as large as the first (§ **174**);

(*ii.*) nine twenty-fifths as large as the first;

(*iii.*) four times as large as the first;

(*iv.*) twice as large as the first.

6. Construct two similar pentagons which shall be to each other as $4 : 9$.

7. Construct two similar regular pentagons, one having three times the perimeter of the other. What is the ratio of their areas?

8. Construct two regular hexagons about the same point as centre, making the less radius of one $= 1\frac{1}{2}$ times that of the other.

9. Construct two regular octagons, making the less radius of one $= \frac{3}{4}$ that of the other. What is the ratio of their perimeters? of their areas?

REVIEW OF CHAPTER VIII.

SYNOPSIS.

1. A *proportion* is an equation between two equal ratios.

2. The truth of a proportion is not affected if the terms of both ratios are transposed.

3. The truth of a proportion is not affected if the extremes are transposed, or if the means are transposed.

4. In every proportion the product of the extremes is equal to the product of the means.

5. Two squares are to each other as the squares of their sides; two parallelograms, or two triangles, as the products of their bases by their altitudes.

6. If two parallelograms, or two triangles, have equal bases, they are to each other as their altitudes; if they have equal altitudes, they are to each other as their bases.

7. *Similar* triangles are triangles that have the same shape.

8. Similar triangles are mutually equiangular, and have their homologous sides proportional.

9. In two similar triangles, the sides that include equal angles have the same ratio.

10. A line drawn through two sides of a triangle parallel to the third side cuts off a smaller triangle similar to the entire triangle.

11. A line drawn through two sides of a triangle parallel to the third side divides those sides proportionally.

12. **I. Law of Similarity.** — Two triangles are similar if they are mutually equiangular.

13. **II. Law of Similarity.** — Two triangles are similar if they have an angle equal, and the sides including this angle proportional.

14. **III. Law of Similarity.** — Two triangles are similar if two sides of one are proportional to two sides of the other, and the angles opposite the greater sides are equal.

15. **IV. Law of Similarity.** — Two triangles are similar if their sides, taken in order, are proportional.

16. Lines through a common point divide parallels proportionally.

17. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, —

(1) each leg is a mean proportional between the hypotenuse and the adjacent segment;
(2) the perpendicular is a mean proportional between the segments of the hypotenuse.

18. The homologous sides of similar triangles are to each other (1) as the corresponding altitudes, (2) as the perimeters.

19. Similar triangles are to each other as the squares of their homologous sides.

20. *Problem.* — To find the fourth proportional to three given lines.

21. *Problem.* — To find the mean proportional between two given lines.

22. *Problem.* — To reduce or enlarge given lines in a given ratio.

23. *Problem.* — To divide a given line into equal parts. (Two ways.)

24. *Problem.* — To measure the distance from an accessible to an inaccessible point.

25. *Problem.* — To measure the distance between two inaccessible points. (Two cases.)

26. *Problem.* — To measure the height of an object. (Three methods.)

27. *Similar* polygons are polygons that have the same shape.

28. Similar polygons are composed of the same number of similar triangles similarly placed.

29. Similar polygons are mutually equiangular, and have their homologous sides proportional.

30. Regular polygons of the same number of sides are similar.

31. The perimeters of similar polygons are to each other as any two homologous sides.

32. Similar polygons are to each other as the squares of their homologous sides.

33. *Problem.* — To construct a polygon similar to a given polygon. (Three methods.)

EXERCISES.

1. A square and a rhombus have equal areas: find the ratio of their perimeters, if the altitude of the rhombus is one-fourth that of the square.

2. What single condition will make two right triangles similar? also two isosceles triangles?

3. Given a triangle, two of whose angles are 55° and 80°, construct a similar triangle one-fourth as large.

4. Given a right triangle, an acute angle of which $= 35^\circ$; construct a similar triangle four times as large.

5. Construct an isosceles triangle with the angle at the vertex $= 30^\circ$, and one side $= 60^{cm}$. Then construct a similar triangle one-ninth as large.

6. Construct a triangle similar to a triangle whose sides are to each other as the numbers 7, 8, 11.

7. If a triangle is to be made three times as large as a given triangle, but similar to it, what values must its angles have? its sides?

8. Prove that parallels divide all lines that intersect them into proportional parts.

Hints.—This is an extension of § **112**. Compare also this theorem with the theorem of § **165**.

9. The sides of a right triangle are 21^m, 28^m, and 35^m; find (*i.*) the segments of the hypotenuse made by a perpendicular let fall from the vertex of the right angle, (*ii.*) the length of this perpendicular.

10. Prove that the bisector of an angle of a triangle divides the opposite side into parts that have the same ratio as the adjacent sides.

Hints.—If ABC is the triangle, BD the bisector, prolong AB till it is met at E by a parallel to the bisector through C. $\triangle ABD \sim \triangle ACE$, and $\triangle CBE$ is isosceles.

11. The three medians of a triangle meet in one point. (A *median* is a line drawn from a vertex and bisecting the opposite side.)

Hints.—Let ABC be the triangle. Draw two of the medians AD and BE meeting at a point O, and then draw $DF \parallel AC$ meeting BE in F. Show that $\triangle DFO \sim \triangle AOE$; whence show that $AO : OD = 2 : 1$. Draw the third median CG, cutting AD in P, and show in like manner that $AP : PD \parallel 2 : 1$; therefore P must coincide with O.

The point O is called the *centre of gravity* of the triangle.

12. Construct a right triangle having given,—

(*i.*) The hypotenuse 70^{cm}, and the ratio $3 : 2$ of the legs.

(*ii.*) The altitude 40^{cm}, and the ratio $3 : 5$ of the legs.

13. Construct on the diagonal of a rectangle an equal rectangle.

14. What is the ratio of the area of a field to that of its plan on paper,—

(*i.*) if the scale of reduction is 1^{cm} to the meter?

(*ii.*) if the scale of reduction is 1^{mm} to the meter?

15. Wishing to find the distance from a point A on the mainland to an island B, I erect at A a line $AC \perp AB$, measure upon it a length $AC = 900^m$, and also measure the angle ACB, which I find to be 80°. From these data find the distance AB.

16. In *Fig. 163*, let $AX = 542^m$, $AY = 735^m$, and the angle $A = 57^\circ\ 30'$; find the distance XY.

17. A vertical rod 5^m long throws upon a horizontal plane a shadow 5.7^m long. At the same time a tower throws a shadow 130^m long. How high is the tower?

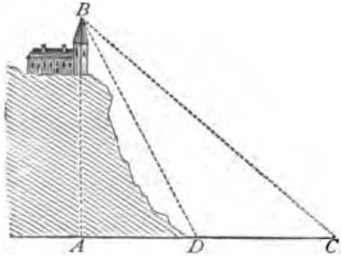

Fig. 174.

18. Explain how to find the height AB (*Fig. 174*) by means of the right triangles ABC and ABD.

19. Explain how, without measuring any angles, the distance XY (*Fig. 175*) can be found by means of observations and measurements performed on this side of the water.

20. The sides of a pentagon are 12^m, 20^m, 11^m, 15^m, and 22^m; the perimeter of a similar pentagon is 16^m; find its sides.

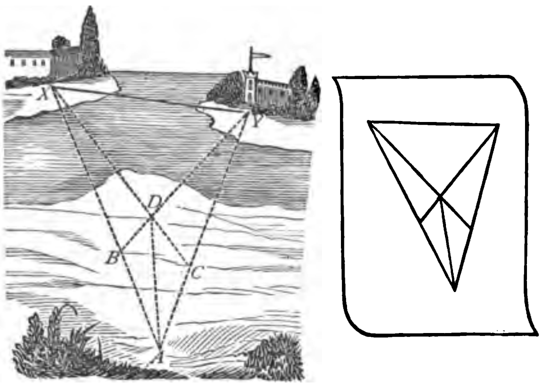

Fig. 175.

21. Construct a polygon similar to a given polygon, and having to it the ratio 4 : 9.

22. Construct a rectangle similar to one given rectangle and equal to another given rectangle.

CHAPTER IX.

THE CIRCLE.

CONTENTS.—I. Sectors, Angles at the Centre, Chords, Segments (§§ 180–184). II. Inscribed Angles (§§ 185–188). III. Secants and Tangents (§§ 189–192). IV. Two Circles (§§ 193, 194). V. Inscribed and Circumscribed Figures (§§ 195–207). VI. Length of a Circumference (§§ 208–211). VII. Area of a Circle (§§ 212–218).

I.—Sectors, Angles at the Centre, Chords, Segments.

§ **180**. Review §§ **26, 27, 28**.

Some new definitions will now be given:—

Definitions.—I. *The parts into which a circle* (*Fig. 176*) *is divided by two radii are called* SECTORS.

II. *The angle between two radii is called an* ANGLE AT THE CENTRE.

III. *A straight line joining two points of a circumference is called a* CHORD.

IV. *A chord passing through the centre is called a* DIAMETER.

V. *A chord divides the circumference into two arcs; if equal, these arcs are called* SEMI-CIRCUMFERENCES; *if unequal, the* GREATER *and the* LESS *arcs, respectively.*

A chord is often said to *subtend* the arcs into which it divides the circumference.

VI. *A chord divides the circle into two parts, called* SEGMENTS; *if equal, they are called* SEMICIRCLES; *if unequal, the* GREATER *and* LESS *segments, respectively.*

In *Fig. 176* point out or name sectors, angles at the centre, chords, a diameter, greater and less arcs, greater and less segments, semi-circumferences, and semicircles.

Two radii form at the centre *two* angles (§ **47**, Note 2); but, unless otherwise stated, the *concave* angle (less than 180°) is always to be understood. To every such angle there *corresponds* a definite arc, a definite chord, a definite sector, and a definite segment. Point out those which correspond to the angle AOB (*Fig. 176*).

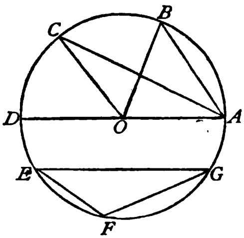

Fig. 176.

From what precedes several properties of the circle follow as corollaries, or by the method of superposition.

Corollaries. — 1. *A circle has only one centre.*

2. *All radii of a circle are equal.*

3. *A diameter is twice a radius of the same circle.*

4. *All diameters of a circle are equal* (why?)

5. *An arc can be placed on an equal arc of the same circle* (*or an equal circle*) *so as to coincide with it.*

6. *Circles are equal whose radii or whose diameters are equal.*

Exercises. — 1. Draw a figure of your own, and illustrate the definitions of this section.

2. Of what is a circumference the locus?

3. What determines (*i.*) the position, (*ii.*) the magnitude, of a circle?

4. What three different positions can a point have as regards the circle?

5. Given a circle, and the distance of a point from the centre; how can you tell whether the point will be inside, in, or outside the circumference?

6. Draw a circle, having given its diameter.

7. In a given circle make a chord equal to a given line.

8. Make a few sectors differing in magnitude.

9. Make two sectors differing in magnitude, but having the same angle at the centre.

10. What arc (greater or less) corresponds to a concave angle at the centre? to a straight angle? to a convex angle? What segment in each of these cases?

11. Make a few segments (greater and less) and name the corresponding arcs and angles at the centre.

12. If the angle at the centre is 200°, what is the corresponding arc? the corresponding segment?

13. If the angle at the centre is 180°, what is the corresponding arc? sector? segment?

14. Divide (free-hand) a sector into two equal parts. Are the parts also sectors?

15. Divide (free-hand) a segment into two equal parts. Are the parts also segments?

16. If you subtract a segment from the corresponding sector, what figure is left?

17. Prove that a diameter is the longest chord in a circle. (Use § **62**.)

§ **181.** **Theorem.**—*In the same circle, or equal circles, to equal angles at the centre correspond equal arcs, chords, sectors, and segments.*

Draw a suitable figure, state hypothesis and conclusions (as concisely as possible in terms of the letters marked on the figure), and prove by superposition.

Remark.—There are altogether four converse theorems. State them, and prove them (by superposition).

Hence it appears that the equality of any one pair of the five elements mentioned in the theorem implies the equality of all the other pairs.

Corollaries.—1. *If the angle at the centre is a straight angle (180°), the corresponding arc is a semi-circumference, and the corresponding sector is a segment equal to a semicircle.*

2. Since the sides of a straight angle form a diameter, *every diameter bisects a circle and also its circumference.*

3. *If the angle at the centre is a right angle, the corresponding arc is half of a semi-circumference, and the corresponding sector is half of a semicircle.*

4. *Two diameters at right angles to each other divide a circle and also its circumference into four equal parts.*

Definition.—*One-fourth of a circle is called a* QUADRANT.

The corresponding arc is the *arc of a quadrant.*

Exercises.—**1.** If the angle at the centre is 180°, what is the arc? the chord? the sector? the segment?

2. How many degrees are there in an arc of a quadrant?

3. What is the ratio of a quadrant and a half to the whole circle?

4. In a given circle make (*i.*) an arc equal to a given arc, (*ii.*) a sector equal to a given sector, (*iii.*) a segment equal to a given segment.

5. Make an arc of 225°.

6. Make an arc equal to twice a given arc.

7. Make an arc equal to five times a given arc.

8. Make a sector equal to twice a given sector.

9. Can you make a segment equal to twice a given segment?

§ **182.** If (*Fig. 177*) the angles AOM, MON, NOP, POB, COQ, QOR, ROD, are equal, then the corresponding arcs and sectors are also equal (§ **181**). And it is obvious (by simple comparison, as in § **37**) that the following ratios hold true:—

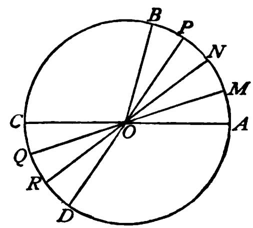

Fig. 177.

Angles $AOB : COD = 4 : 3$;

Arcs $AB : CD = 4 : 3$;

Sectors $AOB : COD = 4 : 3$.

This reasoning may be applied to any angles which have a common measure, and, in higher works on Geometry, is extended to incommensurable arcs with the same result. Hence,—

Theorem.—*In the same circle, or equal circles, two arcs, or two sectors, have the same ratio as the corresponding angles at the centre.*

Corollary.—*A sector has the same ratio to the entire circle that the corresponding arc has to the entire circumference.*

Exercises.—**1.** Make two arcs which shall be to each other in the ratio 2 : 5.

2. What ratios have the following sectors to the entire circle?

(*i.*) A sector of 30°. (*iii.*) A sector of 90°.

(*ii.*) A sector of 75°. (*iv.*) A sector of 325°.

§ **183.** **Theorem I.**—*The diameter perpendicular to a chord bisects this chord and also the arcs subtended by it.*

Proof.—Draw radii to the ends of the chord, and apply § **74.** To prove that the arcs are bisected, make use of § **181** and Axiom III.

What two other properties does this diameter possess?

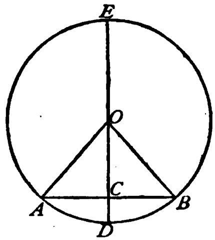

Fig. 178.

Theorem II.—*A perpendicular erected at the middle of a chord passes through the centre of the circle* (*or is a diameter*).

Proof. — This perpendicular passes through every point which is equidistant from the ends of the chord (§ **87**), and the centre of the circle is such a point.

What other properties does this perpendicular have?

Remark.—We see (*Fig. 178*) that the middle of the chord, the points of bisection of the major and minor arcs, and the centre of the circle are in one straight line; further, that this line is perpendicular to the chord; and, finally, that it bisects the two angles at the centre, their corresponding sectors and segments. This makes, in all, eleven conditions which the line CD fulfils (or eleven properties which it possesses). They are so related that, if any two of them are known to be fulfilled, it follows that all the others must also be fulfilled.

Problem.—*To find the centre of a given circle* (*or arc*).

Analysis.—Use the method of loci (§ **92**) in connection with Theorem II. of the present section.

Corollaries.—1. *Through three points not in a straight line a circle can always be drawn.* (Compare Exercise 7, page 102.)

2. *Two circumferences that have three points in common coincide wholly.*

Hence, a circle may be named by three letters at points on its circumference.

Exercises. — 1. Bisect a given arc.

2. Bisect a given sector.

3. Bisect a given segment.

4. Find the centre of a given circle.

5. Find the centre of a given circle, using only one chord and one perpendicular.

6. Find the centre of a circle which shall pass through three given points.

7. Given a curve, how can you ascertain whether it is the arc of a circle or not?

8. Find the locus of the centres of all the circles which pass through two given points.

9. Draw a circle which shall pass through two given points, and whose centre shall be in a given line. (In what case is there no solution? In what case is the problem indeterminate?)

10. Draw a circle with a given radius and passing through two given points. (Two solutions. In what case only one? In what case no solution?)

11. Find the locus of the middle points of a system of parallel chords.

12. Through a given point within a circle draw a chord which is bisected at this point.

§ **184.** Let the radius OA (*Fig. 179*) remain fixed, and conceive that a second radius turns about O as a centre, starting from the position OA, and coming successively into the positions OB, OC, OD, etc. The greater the angle at the centre, AOB, AOC, AOD, etc., the greater the corresponding chord AB, AC, AD, etc. Likewise the greater the chord the nearer it approaches the centre O; and finally, when the moving radius arrives at the position OE, the chord passes through the centre, and becomes a diameter. In this way we may arrive at the following conclusions: —

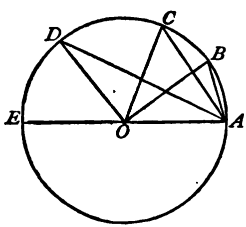

Fig. 179.

Theorems. — I. *The greater the angle at the centre, the greater the corresponding chord (the angle being supposed less than 180°).*

NOTE.—The chord does not, however, increase in the same ratio as the angle at the centre. The chord, for instance, of twice an angle is greater, but not twice as great, as the chord of the angle itself.

II. *The diameter is the longest chord of a circle.*

III. *Equal chords are equally distant from the centre.*

IV. *Of two unequal chords, the greater is nearer to the centre, and conversely.*

Exercise.—Draw through a given point within a circle the longest chord; the shortest chord.

II.—Inscribed Angles.

§ **185.** **Definition.**—*An angle BAC (Fig. 180) whose vertex is in the circumference of a circle, and whose sides are chords, is called an* INSCRIBED ANGLE.

To every inscribed angle corresponds a definite arc and a definite chord. The inscribed angle is often said to "*stand upon*" its arc.

By an *angle incribed in a segment* ($DEFM$) is understood an inscribed angle (DEF) whose sides pass through the ends of the arc of the segment.

Exercises.—**1.** Draw a few inscribed angles, and point out the corresponding arcs and chords.

2. Inscribe several angles in the same segment.

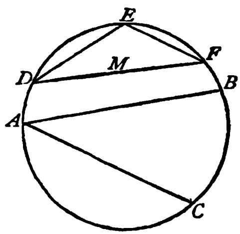

Fig. 180.

§ **186.** **Theorem.**—*An inscribed angle is equal to one-half of the angle at the centre which has the same arc.*

Proof.—The centre of the circle may lie (*i.*) in one of the sides of the angle (*Fig. 181*), or (*ii.*) between the sides of the angle (*Fig. 182*), or (*iii.*) without the sides of the angle (*Fig. 183*). So there are three cases, but the second and third are closely connected with the first.

Case (*i.*).—Draw the radius OC, and use §§ **66** and **80**.

Case (*ii.*). — Draw the diameter AD, apply the proof for Case (*i.*) twice, and add the results.

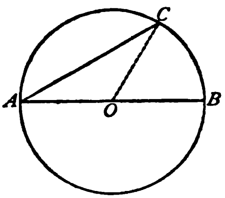

Fig. 181.

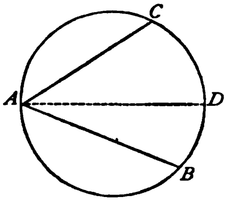

Fig. 182.

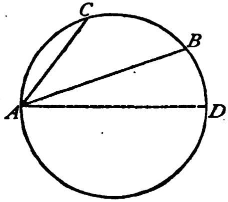

Fig. 183.

Case (*iii.*). — Draw the diameter AD, apply the proof for Case (*i.*) twice, and subtract the results.

Corollaries. — 1. *Inscribed angles which stand upon the same arc are equal.*

Proof. — They are the halves of the same angle at the centre.

2. *All angles inscribed in the same segment are equal.*

Proof. — They are inscribed angles standing upon the same arc; namely, the arc found by subtracting the arc of the segment from the whole circumference.

Upon what arc does the angle DEF (*Fig. 180*), inscribed in the segment $DEFM$, stand?

3. *An angle inscribed in a semicircle is a right angle.*

Proof. — The angle at the centre having the same arc is equal to two right angles.

4. *The sum of the angles inscribed, one in a greater, the other in a less, segment, is equal to two right angles.* Why?

Illustrate by diagrams these four corollaries.

Exercises. — **1.** What is the value of an inscribed angle standing upon an arc of 60°? 153°? 320°?

2. What value has an angle inscribed in a segment whose arc is equal to 90°? 135° 180°? 270°? 300°?

3. Is the angle inscribed in a greater segment greater or less than 90°? What is its value inscribed in a less segment?

4. The arcs included by two parallel chords are equal. (Two methods of proof.)

5. Parallel chords drawn from the ends of a diameter are equal. (Use Corollary 3 of this section, and § **72**.)

6. Find the locus of the vertex of a right angle whose sides pass through two given points.

Hints.—See Corollary 3. Make the line which joins the given points the diameter of a circle.

7. Construct a right triangle, having given its hypotenuse. Is the problem determinate or not?

8. Construct a right triangle, having given the hypotenuse, and the altitude upon the hypotenuse as base. Is this problem determinate or not?

9. Construct a right triangle, having given the middle point of the hypotenuse, the vertex of the right angle, and the length of one leg.

10. Find the locus of the vertex of a given angle whose sides pass through two given points.

11. Find the locus of the middle points of all chords drawn through a given point in the circumference of a given circle. (Use § **183**, Theorem II., and Corollary 3 of the present section.)

§ **187.** If at any point D (*Fig. 184*) of the diameter AB of a circle we erect a perpendicular DC, and join AC and BC, the angle $ACB = 90°$ (§ **186**, Corollary 3); whence it follows, from § **166**, that,

I. $\overline{DC}^2 = AD \times BD$;

II. $\begin{cases} \overline{AC}^2 = AB \times AD; \\ \overline{BC}^2 = AB \times BD. \end{cases}$

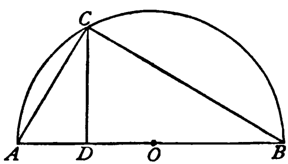

Fig. 184.

Exercises.—**1.** State the above results in general terms as two theorems.

2. Verify the theorems for the case where D is at the centre of the circle. Draw a diagram.

3. If (*Fig. 184*) $OB = 24^{cm}$, and $OD = 10^{cm}$, find DC, AC, and BC.

§ **188.** Through a point P (*Fig. 185*) within a circle draw any two chords APB and CPD. Join AD and CB. Show that $\triangle APD \sim \triangle CPB$ (§§ **54**, **186**, **161**); whence $PA : PD =$

$PC : PB$; or, multiplying together the extremes and the means,

$$PA \times PB = PC \times PD.$$

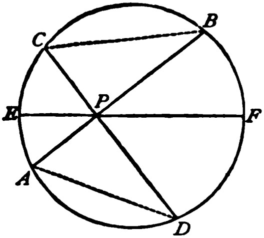

Fig. 185.

Theorem. — *If chords are drawn through any point within a circle, the product of the two parts into which the chord is divided by the point is the same for all the chords.*

Corollary. — Among the chords which may be drawn through P is the diameter EPF; therefore, *the constant value of the above-mentioned product, for a given position of P, is the product of the two parts of the diameter drawn through P.*

Exercises. — **1.** Prove § **187**, Theorem I., using this Corollary and § **183**.

2. If the radius of a circle $= 24^{cm}$, find the product of the parts of any chord drawn through a point 10^{cm} from the centre.

3. Prove that the angle APC (*Fig. 185*), between two intersecting chords AB and CD, is measured by half the arc AEC included between its sides + half the arc DFB included between its sides produced. (Use § **66** and § **186**.)

III. — Secants and Tangents.

§ **189. Definition I.** — *A straight line of indefinite length AB (Fig. 186), which cuts the circumference of a circle in two points is called a* SECANT.

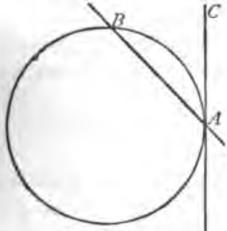

Fig. 186.

If we conceive the secant AB to turn about the point A, towards the right, the other point of intersection B will approach nearer and nearer to A, and will finally coincide with A. At this instant, the secant takes the position CD, and has only one point A in common with the circumference.

Definition II.—*A line which touches a circumference in one point without cutting it is called a* TANGENT.

Such a line is usually said to *touch* the circle; and the point where it touches the circle is called the *point of contact.*

Exercises.—**1.** In what relation does a secant stand to a chord?

2. Can a straight line cut a circle in more than two points?

3. What three different positions may a straight line have with respect to a circle?

§ **190.** Let BC (*Fig. 187*) touch the circle at the point A. Join A to the centre O, and also join any other point of BC, as D, to O. Then $OD > OA$ (why?). That is, OA is shorter than any other line which can be drawn from O to the line BC; therefore (§ **83**), $OA \perp BC$ and $BC \perp OA$.

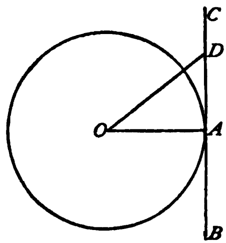

Fig. 187.

Theorem.—*A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

Exercises.—**1.** Find the locus of the centres of circles which touch a given line in a given point.

2. Find the locus of the centres of circles which have a given radius and touch a given line. (Two lines. See § **90.**)

3. The radius perpendicular to a tangent bisects every chord parallel to the tangent.

§ **191.** **Problem.**—*Through a given point to draw a tangent to a given circle.*

There are three cases; for the given point may lie, (*i.*) within, (*ii.*) in, (*iii.*) without, the circumference.

Case (*i.*) cannot be solved. Why?

Case (*ii.*).—§ **190** supplies the means of solution. How? Give the construction.

Case (*iii.*). — *Construction.* — Let A (*Fig. 188*) be the given point, O the centre of the given circle. Join AO, and upon AO as a diameter describe a circle cutting the given circle in B and C. AB and AC are both tangents to the given circle.

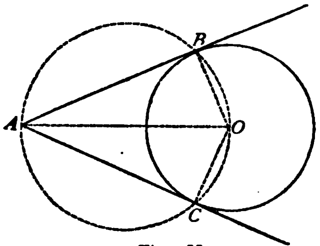

Fig. 188.

Proof. — § **186**, Corollary 3, and § **190**.

Corollary. — *The tangents AB and AC (Fig. 188) are equal.*

Proof. — Show that the triangles ABO and ACO are equal.

Exercises. — **1.** Draw a tangent through a given point in the circumference of a circle.

2. Draw tangents through a given point without the circumference of a circle.

3. If tangents are drawn from a point without a circle, the line which joins the point to the centre of the circle bisects the angle made by the tangents.

4. If the angle between two tangents drawn through a point without a circle is a right angle, find the angle at the centre formed by radii drawn to the points of contact.

5. Draw a circle with a given radius that shall touch a given line in a given point. (Two solutions.)

6. Draw a circle with a given radius that shall touch a given line and pass through a given point (two, one, or no solution, according as the distance from the given point to the given line is $<$, $=$, or $>$ the diameter of the circle).

7. Describe a circle that passes through a given point and touches a given line in a given point. (Where cannot the first point be ?)

8. To a given circle draw a tangent which, —

(*a*) is parallel to a given line.

(*b*) is perpendicular to a given line.

(*c*) makes a given angle with a given line.

9. Describe a circle that touches two given intersecting lines, and, —

(*a*) whose centre is in a given line.

(*b*) which touches one of the lines in a given point.

(*c*) which has a given radius.

10. Describe a circle that touches two given parallel lines, one of them in a given point.

11. Describe a circle that touches two given parallel lines, and passes through a point between them. (Two solutions.)

12. Describe a circle that touches three given lines. (Four, two, or no solution.)

§ **192.** Let (*Fig. 189*) AD be a tangent to a circle at A, and AB any chord drawn from A. Draw the diameter AC, and join B to the centre O of the circle. $CAD = 90°$ (why?) $= \frac{1}{2}$ arc ABC (measured in degrees). $CAB = \frac{1}{2}\, COB$ (why?) $= \frac{1}{2}$ arc CB (measured in degrees). The angle between the tangent and the chord $BAD = CAD - CAB$. Therefore, $BAD = \frac{1}{2}$ arc $ABC - \frac{1}{2}$ arc $CB = \frac{1}{2}$ arc BD.

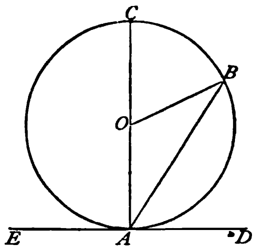

Fig. 189.

Theorem. — *The angle between a tangent and a chord drawn through the point of contact, is measured in degrees by half the arc included between its sides.*

Exercises. — **1.** Prove that the obtuse angle BAE (*Fig. 189*) made by the tangent and the chord is measured by half the arc included between its sides.

2. The angle made by two secants that meet without the circle is measured by half the difference of the arcs included between its sides.

Hint. — Join two alternate points in which the secants meet the circle. Use § **66** and § **130**.

3. The angle made by two tangents drawn through a point without a circle is measured by half the difference of the arcs included between its sides (§ **66** and the Theorem above).

4. The less arc between the points of contact of two tangents drawn through a point without a circle $= 120°$. Find the angle between the tangents.

5. The angle made by a tangent and a secant that meet without the circle is measured by half the difference of the arcs included between its sides.

IV.—Two Circles.

§ **193.** The relative position of two circles depends upon the positions of their centres and the lengths of their radii.

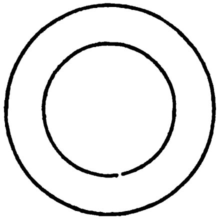

Fig. 190.

Definitions. — I. *Two circles that have the same centre* (*Fig. 190*) *are said to be* CONCENTRIC.

II. *The plane figure lying between their circumferences is called a* RING.

III. *Two circles that have different centres are said to be* ECCENTRIC.

IV. *The line joining their centres is called the* LINE OF CENTRES.

§ **194.** Let (*Fig. 191*) O and P be the centres of two eccentric circles, with the radii r and r' respectively, and let us proceed to examine the different positions which these circles will have, as the distance of their centres $OP = c$ is supposed continually to increase. There are five different cases.

Case (*i.*) (*Fig. 192*). $c < r - r'$. If the distance of the centres is less than the difference of the radii, *the smaller circle lies wholly within the larger.*

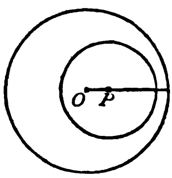

Fig. 191.

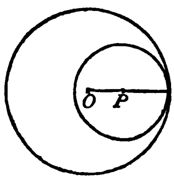

Fig. 192.

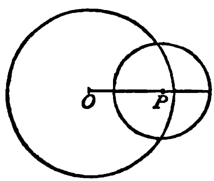

Fig. 193.

Case (*ii.*) (*Fig. 193*). $c = r - r'$. If the distance of the centres is equal to the difference of the radii, *the circles touch, or are tangent, internally.*

Case (*iii.*) (*Fig. 193*). $c > r - r'$ and $< r + r'$. If the distance of the centres is greater than the difference, but less than the sum of the radii, *the circles intersect in two points.*

Case (*iv.*) (*Fig. 194*). $c = r + r'$. If the distance of the centres is equal to the sum of the radii, *the circles touch, or are tangent, externally.*

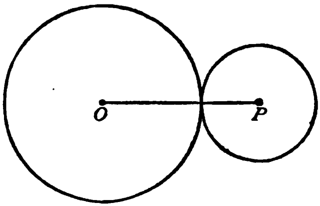

Fig. 194.

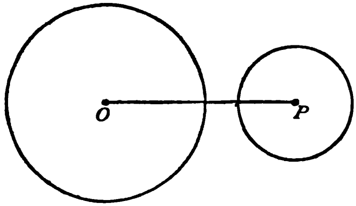

Fig. 195.

Case (*v.*) (*Fig. 195*). $c > r + r'$. If the distance of the centres is greater than the sum of the radii, *the circles lie wholly outside each other.*

Exercises.—**1.** Can you give a reason why two circles cannot meet in more than two points?

2. What different positions can two circles have relatively to each other?

3. Let r and r' be the radii of two circles, c the distance of their centres. What relative positions have the circles for the following values of r, r', and c?

(*a*)	$r = 5$,	$r' = 3$,	$c = 8$.	(*f*)	$r = 6$,	$r' = 6$,	$c = 8$.
(*b*)	$r = 7$,	$r' = 4$,	$c = 2$.	(*g*)	$r = 10$,	$r' = 3$,	$c = 7$.
(*c*)	$r = 6$,	$r' = 2$,	$c = 10$.	(*h*)	$r = 5$,	$r' = 5$,	$c = 10$.
(*d*)	$r = 8$,	$r' = 3$,	$c = 6$.	(*i*)	$r = 12$,	$r' = 7$,	$c = 9$.
(*e*)	$r = 9$,	$r' = 5$,	$c = 4$.	(*k*)	$r = 9$,	$r' = 2$,	$c = 5$.

4. Describe, with the radii 35^{cm} and 25^{cm}, two circles which shall touch each other (*i.*) internally, (*ii.*) externally.

5. In a given circle make two circles such that they shall both touch the given circle internally, and also touch each other externally.

6. With three given radii, m, n, and p, describe three circles which shall mutually touch one another externally.

Hint.—First construct a triangle having the sides $m + n$, $m + p$, and $n + p$.

V.—Inscribed and Circumscribed Figures.

§ **195. Definitions.**—I. *A circle is said to be* Inscribed *in a polygon when its circumference touches all the sides of the polygon; and the polygon is said to be* Circumscribed *about the circle* (*Fig.* 196).

II. *A circle is said to be* Circumscribed *about a polygon when its circumference passes through all the corners of the polygon; and the polygon is said to be* Inscribed *in the circle* (*Fig.* 197).

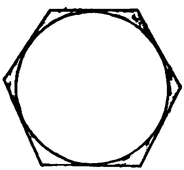

Fig. 196.

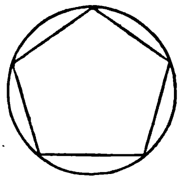

Fig. 197.

In *Fig.* 196, the sides of the polygon are tangents to the circle; in *Fig.* 197, they are chords of the circle.

Note.—In treating this subject we shall first suppose the polygon to be given, and the circle inscribed or circumscribed (§§ **196–201**); and then suppose the circle to be given and the polygon to be inscribed or circumscribed (§§ **202–207**).

§ **196. Problem.**—*To inscribe a circle in a given triangle* ABC (*Fig.* 198).

Construction.—Bisect the angles A and B, and from the intersection O of the bisectors draw $OD \perp AB$. O is the centre of the required circle, OD the radius.

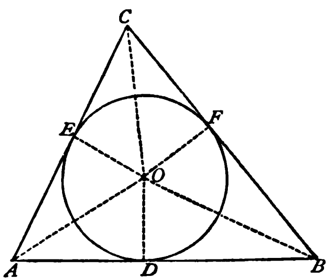

Fig. 198.

Proof.—Draw $OE \perp AC$, $OF \perp BC$, and join OC. Show that $\triangle AOD \cong \triangle AOE$, and $\triangle BOD \cong \triangle BOF$, whence $OD = OE = OF$; therefore, the circle with O as centre and OD as radius touches the sides of the triangle in D, E, and F.

Compare this problem with § **92**, Exercise 8.

Corollary. — *A circle can always be inscribed in a triangle.*

§ **197.** **Problem.** — *To circumscribe a circle about a given triangle.*

Analysis, by means of §§ **87** and **92**.

Draw a figure, and give the construction.

Compare this problem with § **92**, Exercise 7.

Corollary. — *A circle can always be circumscribed about a triangle.*

Exercise. — Prove that if a triangle is equilateral, the radius of the circumscribed circle = twice the radius of the inscribed circle (see § **112**, Exercise).

§ **198.** **Theorem.** — *If a quadrilateral $ABCD$ (Fig. 199) is circumscribed about a circle, the sum of two opposite sides is equal to the sum of the other two sides.*

Proof. — Apply § **191**, Corollary; then add the equal parts of the sides together, obtaining, for the result, $AB + DC = BC + AD$.

Corollary. — *If in a quadrilateral the sum of two opposite sides is equal to the sum of the other two sides, a circle can be inscribed in the quadrilateral.*

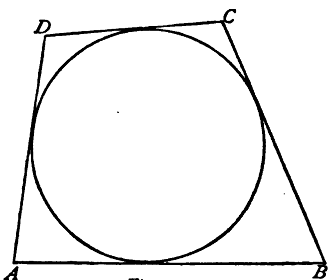

Fig. 199.

Exercises. — **1.** How is the centre of the inscribed circle found?

2. In what two kinds of quadrilaterals can a circle always be inscribed?

3. Draw a rhombus, and inscribe in it a circle.

§ **199.** **Theorem.** — *If a quadrilateral is inscribed in a circle, the sum of the opposite angles is equal to 180°.*

Proof. — Draw a figure of an inscribed quadrilateral, then draw a diagonal, and apply § **186**, Corollary 4.

Corollary. — *If in a quadrilateral the sum of two opposite angles = 180°, a circle can be circumscribed about the quadrilateral.*

Exercises. — **1.** How is the centre of the circumscribed circle found?

2. About what two kinds of quadrilaterals can a circle always be circumscribed?

3. Draw a rectangle, and circumscribe about it a circle.

§ **200.** **Theorem.** — *A circle can always be inscribed in a regular polygon.*

Proof. — Use § **119.** What is the radius of the circle, and what point is its centre?

Exercises. — **1.** Show how to find the centre of the inscribed circle.

2. Inscribe a circle in a regular hexagon.

3. Within a square describe four circles touching each other, and also the sides of the square. What is the common radius of the circles?

§ **201.** **Theorem.** — *A circle can always be circumscribed about a regular polygon.*

Proof. — Use § **119.** What is the radius of the circle, and what point is its centre?

Exercises. — **1.** Show how to find the centre of the circumscribed circle.

2. Circumscribe a circle about a regular hexagon.

3. Circumscribe a circle about a regular octagon.

§ **202.** In order to inscribe regular polygons in circles, or to circumscribe them about circles, it is necessary to divide the circumference of the circle into as many equal parts as there are sides in the polygon.

Problem. — *To divide with ruler and compasses the circumference of a circle into equal parts.*

This problem can be solved only in the following cases: —

Case 1. — Two, four, eight, sixteen, thirty-two, etc., equal parts.

Solution.—A diameter bisects the circumference (§ **181**, Corollary 2); two diameters perpendicular to each other divide it into four equal parts (§ **181**, Corollary 4); repeated bisection of the arcs will divide it into eight, sixteen, thirty-two, etc., equal parts.

Draw a circle, and divide the circumference into two equal parts; four equal parts; eight equal parts.

Case 2.—Three, six, twelve, twenty-four, forty-eight, etc., equal parts.

Solution.—If we construct in a circle (*Fig. 200*) an equilateral triangle OAB, having each side equal to the radius, it is clear that, since $AOB = 60°$, the arc $AB = \frac{1}{6}$ the circumference. Therefore, the radius applied as a chord six times to the circumference will divide it into six equal parts.

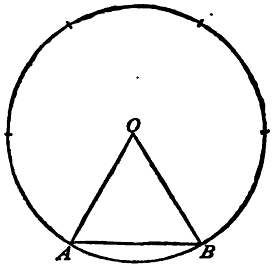

Fig. 200.

The alternate points of division, as A, C, E, divide the circumference into three equal parts (Axiom II.).

How can the circumference be divided into twelve equal parts? into twenty-four? into forty-eight?

Divide a circumference into six equal parts; twelve equal parts.

Case 3.—Five, ten, twenty, forty, eighty, etc., equal parts.

Solution (*Fig. 201*).—Bisect AO in C, and with C as centre, and a radius equal to CD, describe an arc cutting OB in E; OE applied as a chord ten times to the circumference will divide it into ten equal parts, and DE applied as a chord five times will divide it into five equal parts.

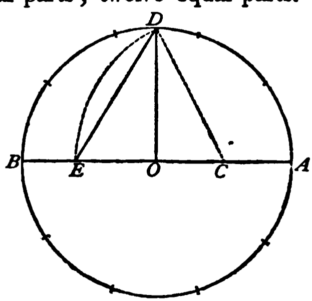

Fig. 201.

The proof that this solution is correct cannot be given here.

How will the division into ten equal parts enable us, without the use of DE, to make the division into five equal parts?

How can a circumference be divided into twenty equal parts?

Divide a circumference into five, and into ten equal parts.

Remark. — With the aid of a protractor (see *Fig. 46*) a circumference may be divided with sufficient exactness for all practical purposes into any number of equal parts. Suppose, for instance, that the number of equal parts is 25; then the angle at the centre corresponding to each part $= \frac{360^\circ}{25} = 14^\circ 24'$. Construct with the protractor, placed so that its centre shall coincide with the centre of the circle, an angle of $14^\circ 24'$; the arc of the given circle contained between the sides of this angle will be $\frac{1}{25}$ the entire circumference.

NOTE. — An angle of 1° (among others) cannot be constructed with the ruler and compasses. The question may here occur: how is a protractor itself graduated into degrees? One way is as follows: divide with ruler and compasses the entire circumference into six equal parts, and then into five equal parts. The difference between two of these parts $(\frac{1}{5} - \frac{1}{6}) = \frac{1}{30}$, the circumference $= \frac{1}{15}$ the semi-circumference. Bisect twice in succession each of these fifteen parts; this divides the semi-circumference into sixty equal parts; finally, each of these sixty parts is trisected by repeated trials, opening or closing a little the compasses till the right opening is obtained. But the most perfect method of graduating a circle into degrees and subdivisions of a degree is by means of nicely-adjusted screw-motion in machines made for this express purpose.

Exercises. — **1.** Find all the angles between 0° and 360° which can be constructed by Case 1.

2. Find all the angles which can be constructed by Case 2.

3. Find all the angles which can be constructed by Case 3.

4. What angle can be trisected with ruler and compasses? Trisect it.

5. Divide with the protractor a circumference into seven, nine, fifteen equal parts.

§ **203.** **Problem.** — *To inscribe in a given circle a regular polygon* (*Fig. 202*).

Construction. — Divide the circumference into as many equal

parts as the polygon has sides, and join in order the points of division by straight lines.

Proof.—The sides AB, BC, etc., are equal, and the angles ABC, BCD, etc., are equal (§ **181**, Remark). Therefore, the inscribed figure is a regular polygon (§ **118**).

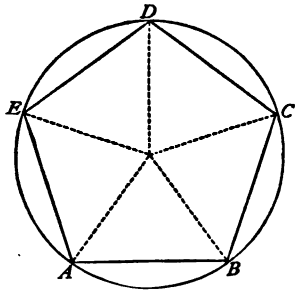

Fig. 202.

Remark.—By means of the following construction, the side of any inscribed regular polygon (except the triangle and the square) can be found to a close degree of approximation. Divide the diameter AB (*Fig. 203*) of the given circle into as many equal parts as there are sides in the polygon to be inscribed. Draw $CD \perp AB$. Prolong AB to E and CD to F, making $AE = DF =$ one of the equal parts. Join EF, and finally join the intersection G of EF with the circumference to the third point of division H of the diameter; GH is the side of the polygon required.

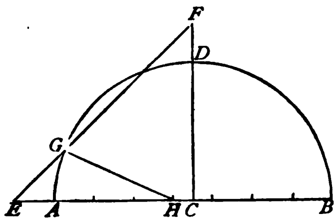

Fig. 203.

Exercises.—Inscribe in a given circle,—

1. A square.
2. A regular pentagon.
3. A regular octagon.
4. A regular polygon of seven sides.
5. A regular polygon of nine sides.
6. A regular polygon of eleven sides.

§ **204. Problem.**—*To circumscribe about a given circle a regular polygon* (*Fig. 204*).

Construction.—Divide the circumference into as many equal parts as the polygon has sides, and draw tangents through all the points of division.

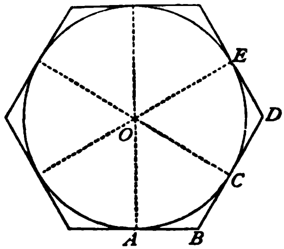

Fig. 204.

Proof. — Show that the quadrilaterals $ABCO$, $CDEO$, etc., are equal (by the method of superposition).

Exercises. — Circumscribe about a given circle, —

1. An equilateral triangle.
2. A regular hexagon.
3. A regular heptagon.
4. A regular nonagon.
5. A regular decagon.
6. A regular dodecagon.

§ **205**. Regular polygons are most easily constructed by employing circles.

Problem. — *To construct a regular polygon, having given the number of sides, and the length AB (Fig. 205) of one side.*

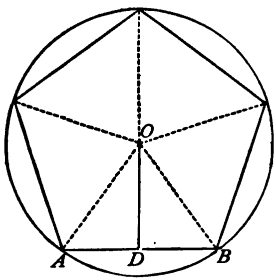

Fig. 205.

Solution. — Find the centre O of the polygon (§ **121**), and about O, with a radius $= OA$, describe a circle; then apply AB to the circumference as many times as there are sides in the polygon.

Remark. — Every regular polygon is composed of a series of equal right triangles (§ **119**, Corollary 2). The sides of one of these triangles, AOD (*Fig.* 205), are the greater radius, the less radius, and the half-side of the polygon. It follows, from § **142**, that these sides cannot have any values which we may choose to give them, but are so related that $\overline{AO}^2 = \overline{AD}^2 + \overline{OD}^2$. The following table gives the values (near enough for most purposes) of the less radius and the side, the greater radius in each case being taken equal to 100.

Polygon of	Greater Radius.	Less Radius.	One Side.
3 sides	100	50	173
4 sides	100	71	141
5 sides	100	81	118
6 sides	100	87	100
7 sides	100	90	87
8 sides	100	92	77
9 sides	100	94	68
10 sides	100	95	62
12 sides	100	96	52

Exercises.—Upon a line 20^{cm} long construct,—

1. A regular pentagon.

2. A regular hexagon.

3. A regular octagon.

4. A regular decagon.

Construct a regular polygon, having given,—

5. Perimeter $= 1.2^{m}$, number of sides $= 8$.

6. Greater radius $= 60^{cm}$, less radius $= 30^{cm}$.

7. Greater radius $= 50^{cm}$, one side $= 26^{cm}$.

8. Less radius $= 27^{cm}$, one side $= 39^{cm}$.

9. Greater radius = one side.

10. One side $= 36^{cm}$, one angle $= 135^{\circ}$.

11. Greater radius $= 45^{cm}$, angle at the centre $= 30^{\circ}$.

12. Less radius $= 32^{cm}$, angle at the centre $= 40^{\circ}$.

13. Since a regular polygon is composed of a series of equal right triangles, any two parts which determine one of these triangles will determine the entire polygon. What are the six parts of one of these triangles? What two parts can always be found if the number of sides is known? How?

14. Give all the cases of two parts that determine a regular polygon.

15. The radius of a circle $= 10^{cm}$. Find the side of the inscribed equilateral triangle; also a side of the inscribed regular decagon.

16. The perimeter of a square $= 40^{cm}$. Find the radii of the inscribed and of the circumscribed circles.

17. In a regular hexagon the line which joins the middle points of two opposite sides is 2^{m} long. Required the perimeter of the hexagon.

18. A square, whose perimeter $= 40^{cm}$, is changed into a regular octagon by cutting isosceles right triangles from its corners. Find the sides of one of these triangles.

19. One side of a marble slab in the shape of a regular octagon $= 42^{cm}$. Find the radius of the inscribed circle.

20. In a circle whose radius $= 15^{cm}$, a regular hexagon and a regular dodecagon are inscribed. Find the difference of their perimeters.

§ **206.** **Problem.**—*Having given the side AB (Fig. 206) of an inscribed regular polygon, to construct the side of an inscribed regular polygon having twice as many sides, and to compute its length.*

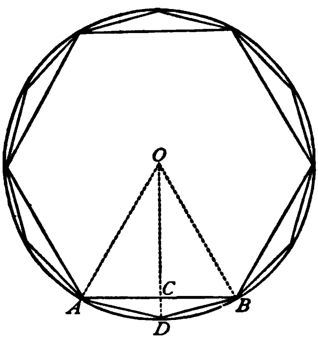

Fig. 206.

Construction. — Bisect the arc AB in D, and join AD. AD is the side of the polygon required.

Proof. — Apply § **181**.

Computation. — If AB and AO are known, then in the right triangle AOC we can compute the value of CO (how?). Subtract CO from DO, and we obtain CD; lastly, in the right triangle ACD, in which AC and CD are known, compute AD (how?).

Exercises. — **1.** In a circle whose radius $= 1^{m}$ a regular hexagon is inscribed. Find the side and the perimeter of the regular inscribed dodecagon.

Solution.—In this case, AB (*Fig.* 206) $= AO = 1^{m}$; $AC = 0.5^{m}$; $CO = \sqrt{\overline{AO}^2 - \overline{AC}^2} = \sqrt{1^2 - 0.5^2} = 0.8660254^{m}$; $CD = DO - CO = 1 - 0.8660254 = 0.1339746^{m}$; $AD = \sqrt{\overline{AC}^2 + \overline{CD}^2} = \sqrt{0.5^2 + 0.1339746^2} = 0.51763818^{m}$. The perimeter of the dodecagon $= 12 \times 0.51763818^{m} = 6.2116582^{m}$.

2. Find the area of the triangle ADB (*Fig.* 206).

3. If the radius of the circle $= 1^{m}$, compute the side and the perimeter of the inscribed regular polygon of twenty-four sides.

The radius of a circle $= 10^{m}$. Find, —

4. The perimeter and the area of the inscribed square.

5. The perimeter and the area of the inscribed regular octagon.

§ **207. Problem.** — *Having given the side AB (Fig. 207) of an inscribed regular polygon, to construct the side of the circumscribed regular polygon having the same number of sides, and to compute its length.*

Construction. — Bisect the arc AB in D; through D draw a tangent, and prolong it until it meets OA and OB prolonged in E and in F. EF is the side of the polygon required.

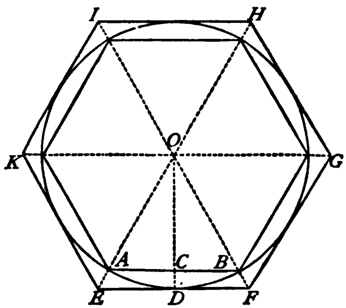

Fig. 207.

Proof — Conceive the same construction carried out for all the equal arcs subtended by the sides of the inscribed polygon; then prove that the triangles OEF, OFG, etc., are equal (by showing that the right triangles which are their halves are all equal).

Computation. — AB and EF are parallel, both being perpendicular to OC; therefore, $\triangle AOB \sim \triangle EOF$, and $OC : OD = AB : EF$. In this proportion, OD and AB are given, and OC can be found from the right triangle AOC (how?). EF can then be computed (how?).

Exercises. — **1.** In a circle whose radius $= 1^m$ a regular hexagon is inscribed. Find the side and the perimeter of the regular circumscribed hexagon.

Solution. — If (*Fig. 207*) $OD = 1^m$, then $AB = 1^m$, and $OC = \sqrt{\overline{OA}^2 - \overline{AC}^2} = \sqrt{1^2 - 0.5^2} = 0.8660254^m$; hence $0.8660254 : 1 = 1 : EF$. $EF = 1.1547005^m =$ one side of the circumscribed polygon. And the perimeter $= 6 \times 1.1547005^m = 6.928203^m$.

2. In a circle whose radius $= 1^m$ the side of the inscribed regular dodecagon has been found (§ **206**) equal to 0.5176318^m. Find the side and the perimeter of the circumscribed regular dodecagon.

The radius of a circle $= 10^m$. Find, —

3. The perimeter and the area of the circumscribed square.

4. The perimeter and the area of the circumscribed regular octagon.

VI.—Length of a Circumference.

§ **208**. In order to find the length of a circumference, we make use of two general truths, which are sufficiently obvious from what precedes.

I. *The perimeter of an inscribed regular polygon is always less, and that of a circumscribed regular polygon is always greater, than the length of the circumference.*

II. *The greater the number of sides, the nearer the perimeter in either case approaches to the length of the circumference.*

If the radius of the circle = 1, the perimeter of the inscribed regular hexagon = 6 (§ **202**, Case 2), and that of the circumscribed regular hexagon = 6.928203 (§ **207**); therefore, the length of the circumference is more than 6, but less than 6.928203, times the length of the radius.

If now we inscribe and circumscribe regular polygons of twelve sides, twenty-four sides, forty-eight sides, etc., and compute to the sixth decimal place the values of their perimeters, we shall obtain the following results: —

Number of Sides in the Polygon.	Perimeter of the Polygon.	
	Inscribed.	Circumscribed.
6	6.000000	6.928203
12	6.211658	6.430782
24	6.265257	6.319320
48	6.278700	6.292172
96	6.282066	6.285430
192	6.282905	6.283746
384	6.283115	6.283325
768	6.283168	6.283220
1536	6.283181	6.283194
3072	6.283183	6.283187

Since the length of the circumference always lies between the perimeters of the inscribed and circumscribed polygons having the same number of sides, and the perimeters of these polygons having 3072 sides agree in value in the first five decimal places, therefore the number 6.28318 expresses the length of the circumference correctly to five decimal places; that is, the length of the circumference is 6.28318 times the length of the radius, or 3.14159 times the length of the diameter.

The number which expresses the ratio of the circumference of a circle to its diameter, is often denoted, for the sake of convenience, by the Greek letter π[1]; that is, $\pi = 3.14159$, very nearly.

For most practical purposes we may take $\pi = 3\frac{1}{7} = \frac{22}{7}$, or, decimally, $\pi = 3.14$. When greater accuracy is required, more decimal places must be used. If more than five decimals are desired, they may be obtained by continuing the above process, or much more easily by means of other methods known to mathematicians.

But whatever method be used, the exact value of the ratio of the circumference of a circle to the diameter cannot be found. The farther the computation is carried, the nearer we approximate to the true value, but the series of decimals which we obtain never comes to an end. This amounts to saying that the circumference of a circle and its diameter are two incommensurable lines (§ **38**).

NOTE.—The problem of finding the ratio between the circumference of a circle and its diameter exercised the minds of geometers in very early times. In an ancient Brahmin work, called *Ayeen Akbery*, the value is assigned as $\frac{3927}{1250} = 3.1416$. Archimedes, the greatest geometer of antiquity (died 212 B.C.), using the method of inscribed and circumscribed polygons, up to polygons of 96 sides, found that the value lay between $3\frac{1}{7}$ and $3\frac{10}{71}$. Ludolph of Cologne, with prodigious labor, carried the computation to polygons of 32,212,254,720 sides, which gives the value correct to 35 decimal places. Metius found it to be $\frac{355}{113}$, which is correct to six decimal places. And Professor Richter of Elbing, Germany, has computed the value of π to 500 decimal places!

But there is no practical advantage in obtaining more than the first five or six places; the value $\pi = 3.14159$ is so near the truth that the error made in computing

[1] π (pronounced like *p*) is the first letter of a Greek word meaning *circumference.*

the circumference of a circle with a diameter of one mile is considerably less than one inch. Accordingly, this problem, which engaged the attention of geometers so deeply when their methods of approximation were less perfect, has now sunk to the rank of those useless questions over which only persons ignorant of geometrical science are willing to waste their time.

§ **209**. Let r, d, and c denote the numbers which express, in terms of the same unit, the lengths of the radius, the diameter, and the circumference, respectively, of a circle; then, from what precedes, —

$$\mathbf{c} = \pi \mathbf{d} = 2\pi \mathbf{r}. \qquad [10.]$$

Read this formula in words. If r is given, how can c be found? If c is given, how can r and d be found?

For example, if $r = 4^{\text{m}}$, then $d = 8^{\text{m}}$, and $c = 8 \times 3.14 = 25.12^{\text{m}}$. Conversely, if $c = 20^{\text{m}}$, then $d = \frac{20}{3.14} = 6.36^{\text{m}}$, and $r = 3.18^{\text{m}}$.

It is obvious, from Formula [10], that if either r or d be doubled, trebled, quadrupled, etc., c will be doubled, trebled, quadrupled, etc.; in other words, —

The circumferences of two circles are to each other as their radii, or as their diameters.

§ **210**. An arc may be measured in degrees, minutes, and seconds (angular measure), or its length may be expressed (like that of any line) in meters, feet, or other linear units.

It follows, from § **181**, that, —

Length of an arc : *whole circumference* = *corresponding angle at the centre* : *360°*.

By means of this proportion such questions as the following may be solved: —

1. In a circle whose radius $= 5^{\text{m}}$, find the length of an arc of $45°$.

The circumference $= 10 \times 3.14 = 31.4^{\text{m}}$. Length of arc of $45°$: $31.4 = 45° : 360°$. Length of the arc $= \frac{31.4 \times 45}{360} = 3.92^{\text{m}}$.

2. In the same circle, how many degrees, etc., are there in an arc 6^{m} long?

The above proportion in this case becomes, $6 : 31.4 =$ No. of degrees in the arc : $360°$. No. of degrees in the arc $= \frac{6 \times 360}{31.4} = 68° \, 47' \, 23''$.

§ 211. Practical Exercises.

(*When not otherwise specified, take* $\pi = 3\frac{1}{7}$.)

Find the circumference of a circle, if the radius is equal to,—

1. 7^{m}. **4.** $6\frac{3}{17}^{ft}$. **7.** 3.27^{m} ($\pi = 3.14$).
2. 236^{m}. **5.** 8^{in}. **8.** 75^{cm} ($\pi = 3.14159$).
3. 4.38^{km}. **6.** $\frac{1}{8}^{mile}$. **9.** 8425^{km} ($\pi = 3.1415926$).

Find the radius and the diameter of a circle, if the circumference is equal to,—

10. 17^{m}. **13.** $15\frac{1}{4}^{ft}$. **16.** 9.74^{m} ($\pi = 3.14$).
11. 162.4^{m}. **14.** 11^{in}. **17.** 64^{cm} ($\pi = 3.14159$).
12. 2.16^{km}. **15.** 1^{mile}. **18.** 40^{km} ($\pi = 3.1415926$).

19. How many revolutions does a car-wheel, the diameter of which $= 1.4^{m}$, make in going a distance of 132^{km}?

Ans. Circumference of wheel $= \frac{1.4 \times 22}{7} = 4.4^{m}$; No. of revolutions $= \frac{132,000}{4.4} = 30,000$.

20. If the driving-wheels of a locomotive have a radius of 3^{ft}, how many revolutions will they make in going from New York to Boston, a distance of 236 miles?

21. The diameter of a wheel $= 75^{cm}$, and in going a certain distance it was observed to make 3000 revolutions. Required the distance.

22. What is the diameter of a circular reservoir, if a man in walking around it makes 840 paces, and each pace $= 0.625^{m}$?

23. What must be the diameter of a round dining table for eight persons, if 77^{cm} is allowed to each person?

24. How many nails are required to fasten the cloth covering about the edge of a circular table 1.4^{m} in diameter, the distance between two nails to be 4^{cm}?

25. A toothed wheel 3.756^{m} in diameter has 360 teeth. Find the distance between the centres of two teeth.

26. On the circumference of a wheel are 36 teeth, and the distance between the centres of two teeth $= 18^{mm}$. Find the diameter of the wheel.

27. The radii of three concentric circles are 1, 2, and 3^{m}. Find their circumferences.

28. The diameters of two concentric circles are 12^{cm} and 20^{cm}. Find the diameter of a circle whose circumference lies half-way between the other two.

29. The diameters of two concentric circumferences are 8^{cm} and 10^{cm}. Find the circumference lying midway between them.

30. If the diameters of the ends of a smooth log are 3^{dm} and 5^{dm}, what is the diameter of a section midway between the ends?

31. Find the exterior and interior circumferences of a circular tank, the exterior diameter being 1.08^{m} and the thickness of the side 3^{cm}.

32. A circular reservoir is dug 9.6^{m} in diameter, 7.2^{m} deep. It is then lined with stones 28^{cm} long, 12^{cm} wide, 8^{cm} high. Find the number of stones required, allowing 1^{cm}, outside measurement, between the stones for mortar.

33. An iron wheel 1.26^{m} in diameter has teeth upon its circumference, distant $.6^{cm}$ from centre to centre; (*i.*) how many teeth are there, and (*ii.*) what is the length of the circumference in which the highest points of the teeth lie, the height of the teeth being 7^{cm}?

34. If the length of the minute-hand of a watch $= 18^{mm}$, what distance does its end pass over in one day? in one year?

35. What distance is traversed by a point in the circumference of a water-wheel 2^{m} in diameter, if the wheel makes 1400 revolutions in one hour?

36. The earth turns on its axis once a day. If its diameter $= 8000$ miles, find the velocity per second of a point on the Equator.

37. What ought to be the diameter of a millstone which is to make 100 revolutions per minute, if experience teaches that to secure good grinding a point on the circumference should have a velocity of about 8^{m} per second?

38. Draw two concentric circles with the radii 25^{cm} and 40^{cm}; then describe a concentric circle whose circumference equals the sum of the two other circumferences. Find, also, its length.

39. What is the radius of a circle whose circumference is equal to the difference between the circumferences of circles 24^{cm} and 45^{cm} in diameter?

40. Find the diameter of a circle whose circumference is equal to five times that of a circle 30^{cm} in diameter.

41. Describe two circles, making the circumference of one equal to one-quarter that of the other.

42. The circumferences of two concentric circles are 50^m and 80^m. How far apart are they?

43. Of two toothed wheels which work together, the first has fifty, the second eighty teeth. How many revolutions will the first make while the second makes sixty? What diameter has the first, if that of the second $= 1.32^m$?

Hint.—The distance between the centres of the teeth on both wheels must be the same.

44. The diameter of a toothed wheel having 100 teeth $= 60^{cm}$. How many teeth must a wheel have, which, when connected with the first wheel, makes ten revolutions while the first makes one? And what must be its diameter?

45. Of two belted wheels in a machine, the first makes twelve revolutions while the second makes five. If the diameter of the first wheel $= 96^{cm}$, find that of the second.

46. Invent and describe an arrangement of toothed wheels, such that, if the first wheel revolves once a day, the last wheel will revolve once a second.

47. The hind wheels of a carriage are 1.5^m in diameter. How many revolutions will they make while the fore wheels, 1^m in diameter, make 2500 revolutions?

48. In a circle 2^m in diameter find the length of the arc subtended by one side of,—

(*i.*) The inscribed square.
(*ii.*) The inscribed regular pentagon.
(*iii.*) The inscribed regular octagon.
(*iv.*) The inscribed regular decagon.

49. Diameter of a circle $= 3.3^m$. How long is an arc of $140°$?

50. What is the diameter of the circle if an arc of $40°$ has a length of 8.3^m?

51. What is the radius of a circle in which each degree of the circumference has a length of 1^{cm}?

52. What is the length of $1°$ on a protractor, the diameter of which $= 48^{cm}$?

53. Find the ratio between the lengths of degrees on two concentric circles, the radii of which are 1^{dm} and 8^{dm}.

54. How many degrees, etc., are there in an arc equal in length to the radius of the circle?

55. Find the ratio between an arc of a quadrant and the corresponding chord.

VII.—Area of the Circle.

§ **212**. The area of a circle is always less than that of a circumscribed regular polygon, and greater than that of an inscribed regular polygon; moreover, the areas of these polygons approach nearer to each other, and to the area of the circle, as the number of their sides is increased. We might, therefore, find the area of a circle by a process similar to that employed for finding the length of the circumference. The following method, however, is much better: —

Circumscribe a regular polygon about a circle (as in *Fig. 204*), and let $r =$ its less radius $=$ radius of the circle. Now it is clear, that by increasing the number of sides in the polygon, we can make its area approach the area of the circle as nearly as we please, but we cannot make it absolutely equal to the area of the circle; in other words, —

$$\text{Area of the circle} = \begin{cases} \text{that value which the area of the polygon} \\ \text{approaches, but never absolutely reaches, as} \\ \text{the number of its sides is increased.} \end{cases}$$

This value of the area of the polygon is termed its *limiting* value, or simply its LIMIT; so that we may write the above equation more briefly, thus: —

$$\text{Area of the circle} = \text{limit of area of the polygon.}$$

By § **133**, area of the polygon $= \frac{1}{2} r \times$ its perimeter. This equation is true, *no matter how great or how small* the number of sides. Now, by increasing the number of sides, the perimeter can be made to approach the circumference of the circle as nearly as we please, but can never quite reach it. Therefore, —

$$\text{Limit of area of the polygon} = \tfrac{1}{2} r \times \text{circumference of the circle.}$$

Therefore (Axiom I.), —

$$\textit{Area of circle} = \tfrac{1}{2} r \times \textit{its circumference.}$$

§ **213.** If we substitute for the circumference the values πd and $2\pi r$, we obtain the very useful formula, —

$$\textbf{Area of a circle} = \tfrac{1}{4}\pi d^2 = \pi r^2. \qquad [11.]$$

For example: if the radius $r = 8^{m}$, the area $S = 3.14 \times 64 = 200.96^{qm}$.

From Formula [11] we find that the values of d and of r in terms of S and π are, —

$$d = \sqrt{\frac{4S}{\pi}} = 2\sqrt{\frac{S}{\pi}}, \qquad r = \sqrt{\frac{S}{\pi}}.$$

Read these formulas in words.

For example: if $S = 20^{qm}$, then $r = \sqrt{\dfrac{20}{3.14}} = \sqrt{6.37} = 2.52^{m}$.

In Formula [11] assume any value for r (or d), then double it, treble it, quadruple it, etc., and work out for each case the corresponding value of the area of the circle. We find that the areas are to each other as the numbers 1, 4, 9, 16, etc. That is, —

The areas of circles are to each other as the squares of their radii (or their diameters).

§ **214. Problem.** — *To transform a given circle into a square.*

By *computation.* Find the area of the circle, extract its square root, and the result will be one side of the square required.

Let a = the radius of the given circle; then its area $= \pi a^2$, and the side of the equivalent square $= \sqrt{\pi a^2} = a\sqrt{\pi} = a\sqrt{3.14159} = a \times 1.77245$.

By *construction.* Draw a straight line equal to half the circumference, and then find, by § **170**, a mean proportional between this line and the radius; this will be a side of the required square.

Remark. — A square which shall be exactly equivalent to a given circle cannot be found, because the value of π cannot be exactly found. In other words, the problem of "squaring the circle" admits only of an approximate solution. But the value

$\pi = 3.14159$ gives a result far more accurate than could be actually constructed with ruler and compasses.

Exercises.—**1.** Draw a circle, and then find the side of an equivalent square, (*i.*) by computation, (*ii.*) by construction. How do the results agree?

2. Construct a square, and then make a circle equivalent to the square.

3. Find the ratio of the areas of a square and a circle, if,—

(*i.*) A side of the square is equal to the radius of the circle.

(*ii.*) A side of the square is equal to the diameter of the circle.

§ **215.** Upon the sides of the right triangle ABC (*Fig. 208*), as diameters, describe circles. By § **142**,—

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

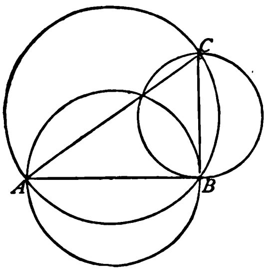

Fig. 208.

If we multiply the terms of this equation by $\frac{1}{4}\pi$, the equality is not destroyed (Axiom IV.), and we obtain,—

$$\tfrac{1}{4}\pi \times \overline{AC}^2 = \tfrac{1}{4}\pi \times \overline{AB}^2 + \tfrac{1}{4}\pi \times \overline{BC}^2.$$

Now these three quantities, taken in order, are the areas of the circles described upon the hypotenuse and the two legs as diameters.

Theorem.—*The area of the circle described upon the hypotenuse of a right triangle as diameter is equal to the sum of the areas of the circles described upon the two legs as diameters.*

NOTE.—This is a special case of a very general theorem, which may be called the Generalized Theorem of Pythagoras, namely: *if upon the sides of a right triangle we construct any three similar figures, the area of the figure constructed upon the hypotenuse will be equal to the sum of the areas of the figures constructed upon the legs.*

Exercises.—**1.** Make a circle equal to the sum of two given circles.

2. Make a circle equal to twice a given circle.

3. Make a circle equal to the difference of two given circles.

4. Make a circle equal to half of a given circle.

§ **216.** **Problem.**—*To find the area of a circular sector AOB (Fig. 209), having given the radius of the circle and the angle of the sector.*

Solution.—From § **182**, Corollary, and § **212**, it follows that the area of a sector is found by *multiplying its arc by half the radius of the circle.*

Example.—Let the radius $= 14^{m}$, and the angle of the sector $= 40°$. Then,—

Circumference of circle $= 2 \times 3\frac{1}{7} \times 14$.

Arc of sector $= \frac{40}{360} \times$ circumference.

Area of sector $=$ its arc $\times 7 = 68.44^{qm}$.

Compute the sector for the following values of the angle: $90°$, $180°$, $270°$.

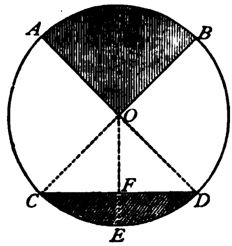

Fig. 209.

§ **217.** **Problem.**—*To find the area of a circular segment CED (Fig. 209), having given the radius of the circle, the angle at the centre COD, and the height EF of the segment.*

Solution.—Compute the areas of the sector COD and the triangle COD. Then, segment $CED =$ sector $COD -$ triangle COD.

Example.—Let the radius $= 1^{m}$, the angle $COD = 40°$, and the height $EF = 6^{cm}$. Then (§ **216**), the sector $COD = \frac{40\pi}{360} = \frac{\pi}{9} = 0.349^{qm}$. In the triangle COD, the altitude $OF = OE - EF = 0.94^{m}$; by § **187**, I., half the base $CF = \sqrt{EF(2 - EF)} = 0.342^{m}$; the base $CD = 0.684^{m}$; and the area of the triangle $= \frac{0.94 \times 0.684}{2} = 0.32148^{qm}$. Therefore, the area of the segment $= 0.349 - 0.32148 = 0.02752^{qm}$; an answer correct enough for all practical purposes.

NOTE.—The height EF (*Fig.* 209) of a segment, and the angle COD at the centre, are not independent of each other, but are connected by a relation which it is the business of Trigonometry to investigate.

§ 218. Practical Exercises.

(*Take* $\pi = 3\frac{1}{7}$, *unless otherwise stated.*)

Find the area of a circle, having given, —

1. The diameter $= 11^{m}$.

2. The diameter $= 4.37^{m}$.

3. The diameter $= 9^{m}$.

4. The radius $= 37^{cm}$.

5. The radius $= 3\frac{1}{7}^{ft}$.

6. The radius $= 4\frac{5}{8}^{miles}$.

Find the area of a circle ($\pi = 3.14159$), having given, —

7. The diameter $= 8.346^{m}$.

8. The diameter $= 16.37^{km}$.

9. The radius $= 27.19^{m}$.

10. The radius $= 5\frac{1}{4}^{ft}$.

Find the radius of a circle, having given, —

11. The area $= 100^{qm}$.

12. The area $= 946^{qm}$.

13. The area $= 23.37^{qm}$.

14. The area $= 2000^{sq.\,ft}$.

15. The area $= 682.4^{sq.\,in}$.

16. The area $= 72^{sq.\,miles}$.

17. The radius of a circle $= 7^{m}$. Find the area, (*i.*) taking $\pi = 3\frac{1}{7}$, (*ii.*) $\pi = 3.14159$. What is the difference (in square centimeters) of the results? Which result is more correct?

18. The diameter of a circular park is 1^{km}. Find the difference in its areas computed, (*i.*) taking $\pi = 3\frac{1}{7}$; (*ii.*) taking $\pi = 3.14159$.

19. A man buys a circular field, diameter $= 1600^{ft}$, for four cents per square foot. What difference will there be in the price paid for the field according as π is taken equal to $3\frac{1}{7}$ or equal to 3.14159?

20. Describe a circle, and then find its area.

21. Find the area of a circle if its circumference $= 80^{m}$.

22. The circumference of a tree measures 2.6^{m}. Find the area of a cross-section.

23. If the length of a circular race-course is to be 1^{mile}, how many acres of land will it cover?

24. If the circumference of a garden-bed $= 48.25^{m}$, what is its area?

25. Around the banks of a circular pond are planted 256 poplars, 6^{m} apart. Find the area of the pond.

26. What will it cost to cover a round table the radius of which $= 75^{cm}$, if the cloth is 60^{cm} wide and costs $2.00 per meter?

Find the radius, the circumference, and the area of a circle equal to, —

27. The sum of two circles whose diameters are 5^{m} and 6^{m}.

28. The sum of two circles whose areas are 60^{qm} and 80^{qm}.

29. The difference of two circles whose radii are 3^{m} and 4^{m}.

30. The difference of two circles whose circumferences are 33^{m} and 44^{m}.

Find the radius of a circle,—

31. Three times as large as a circle whose radius $= 7^m$.

32. Five and one-half times as large as a circle whose diameter $= 12.6^m$.

33. One-ninth as large as a circle whose circumference $= 45.23^m$.

34. If a horse, tied by a rope 4^m long to a stake driven in the ground, eats all the grass which he can reach in three hours, how long must the rope be in order that the grass may last twelve hours?

35. If the diameters of the holes in the side of a tank are as $3 : 1$, how much more water will flow in an hour out of one than out of the other?

36. If 80 flowers will grow on a circular garden-bed, and I wish another bed on which 800 of the same flowers will grow, what diameter must it have compared with that of the first bed?

37. Find the area of a circular ring (*Fig. 209*), the radii of the circles being 6^m and 7^m.

38. Find the area of a ring, the two circumferences being 20^m and 30^m.

39. Describe two concentric circles; then find the area of the circular ring.

40. In the middle of a circular pond 400^m in diameter stands a circular island 90^m in diameter. Find the area of the surface covered by the water.

41. Find the area of a ring if the outer circumference $= 85^{cm}$ and the breadth of the ring $= 10^{cm}$.

42. What will it cost to have a walk 1^m wide about a circular reservoir 64^m in diameter, the price of paving being $2.50 per square meter?

43. A target consists of a black circle 0.75^m in diameter, surrounded by a white ring 0.28^m wide. Find the total area covered by the target.

44. The outer and inner circumferences of a round tower are 17.2^m and 12.8^m. Find the ground area covered by the wall.

45. How wide is a ring if the areas of the two circles are 200^{qm} and 64^{qm}?

46. If a ring is to be 4^m wide, and is to contain 80^{qm}, what must be the radii of the two circles?

47. To the shorter sides of a table, 2^m by 1.25^m, semicircular leaves are added. Find the total area of the table.

48. Find the total area of the figure formed by adding semicircles to the four sides of a square, one side of which $= 1.2^m$.

49. A square, a rectangle, and a circle have the same perimeter, 8^m. Find the three areas, the rectangle being two-thirds as wide as it is long.

50. The circumference of a circle is equal to the perimeter of a square whose side is 10^m. Find the difference in their areas.

NOTE.—Of all plane figures with equal perimeters the circle has the greatest area.

51. A circle and a square have the same area, 120^{qm}. Find the difference between the circumference of the circle and the perimeter of the square.

52. If a circle, a square, a regular hexagon, and a regular octagon each contain 10.4^{a}, find the differences between the perimeters of the polygons and the circumference of the circle.

53. A circular piece of lead is recast in the shape of a square, the thickness remaining the same. If the radius of the circle $= 45^{cm}$, find a side of the square.

54. Find the diameter of a circle equivalent to a regular octagon whose side $= 2^{m}$.

55. The radius of a circle $= 8^{m}$. Find the difference between its area and the area of the inscribed regular hexagon.

56. In a picture-frame 60^{cm} square is a circular picture, the radius of which $= 20^{cm}$. Find the area of the part not covered by the picture.

57. In a circle whose radius $= 30^{cm}$, find the area of a sector of 80°.

58. If the radius of a circle $= 76^{cm}$, find the area of a sector whose arc is 20^{cm} long.

59. How large is a sector if its arc contains 120° and is 40^{cm} long?

60. What part of the entire circle is the sector whose arc equals the radius of the circle?

61. If the area of a sector of $45^{\circ} = 2^{qm}$, find the radius of the circle.

62. The radius of a circle $= 3^{m}$, the height of a segment $= 1.968^{m}$, and the angle of the segment $= 98^{\circ}$. Find the area of the segment.

In a circle whose radius $= 4^{m}$, find the area of the segment cut off by the side of, —

63. The inscribed equilateral triangle.

64. The inscribed square.

65. The inscribed regular hexagon.

66. The inscribed regular octagon.

REVIEW OF CHAPTER IX.

SYNOPSIS.[1]

I.—Sectors, Angles at the Centre, Chords, Segments.

§ **180**. Review of §§ **26–28**. Definitions of a *sector*, an *angle at the centre*, a *chord*, a *diameter*, a *semi-circumference*, *greater* and *less* arcs, a *segment*, *greater and less* segments. Six *corollaries*.

§ **181**. Equal angles at the centre, their arcs, chords, sectors, and segments. Five *theorems*. Four *corollaries*. Definition of a *quadrant*.

§ **182**. Unequal angles at the centre, their arcs and sectors. One *theorem*. One *corollary*.

§ **183**. Properties of a diameter perpendicular to a chord. Two *theorems*. One *problem*. Two *corollaries*.

§ **184**. Equal and unequal chords, their angles at the centre, and their distances from the centre. Four *theorems*.

II.—Inscribed Angles.

§ **185**. Definition of an *inscribed* angle.

§ **186**. Measure of an inscribed angle. One *theorem*. Four *corollaries*.

§ **187**. Application of § **166** to the circle. Two *theorems*.

§ **188**. Product of the parts of a chord passing through a point within a circle. One *theorem*. One *corollary*.

III.—Secants and Tangents.

§ **189**. Definitions of a *secant* and a *tangent*.

§ **190**. Relation between a tangent and a radius drawn to the point of contact. One *theorem*.

§ **191**. Tangent to a circle through a given point. One *problem*. One *corollary*.

§ **192**. Measure of the angle between a tangent and chord. One *theorem*.

IV.—Two Circles.

§ **193**. Definition of *concentric* and *eccentric* circles, a *ring*, the *line of centres*.

§ **194**. Relative positions of two eccentric circles. Five cases.

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[1] Here, and in the remaining chapters, the synopsis is presented in a condensed form which may be made the basis of a more or less extended review according to the discretion of the teacher.

V.—Inscribed and Circumscribed Figures.

§ **195.** Definitions of *inscribed* and *circumscribed* figures.
§ **196.** Circle inscribed in a triangle. One *corollary.*
§ **197.** Circle circumscribed about a triangle. One *corollary.*
§ **198.** Relation between the sides of a circumscribed quadrilateral. One *theorem.* One *corollary.*
§ **199.** Relation between the angles of an inscribed quadrilateral. One *theorem.* One *corollary.*
§ **200.** Circle inscribed in a regular polygon. One *theorem.*
§ **201.** Circle circumscribed about a regular polygon. One *theorem.*
§ **202.** Division of a circumference into equal parts. Three cases.
§ **203.** Regular polygon inscribed in a circle.
§ **204.** Regular polygon circumscribed about a circle.
§ **205.** Construction of regular polygons.
§ **206.** An inscribed regular polygon, and the inscribed regular polygon with twice as many sides.
§ **207.** An inscribed regular polygon, and the circumscribed regular polygon with the same number of sides.

VI.—Length of a Circumference.

§ **208.** The general truths which enable us to find the length of a circumference. Method of applying these truths. Value of π.
§ **209.** Formula for the length of a circumference. Relation between two circumferences, their radii, and their diameters.
§ **210.** Two ways of estimating the length of an arc.
§ **211.** Practical exercises.

VII.—Area of a Circle.

§ **212.** Method of finding the area of a circle.
§ **213.** Formula for the area of a circle. Relation between the areas of circles, their radii, and their diameters.
§ **214.** A circle and the equivalent square.
§ **215.** Relation between the areas of circles described upon the sides of a right triangle as diameters.
§ **216.** Area of a sector.
§ **217.** Area of a segment.
§ **218.** Practical exercises.

EXERCISES.

1. In a circle, as the angle at the centre increases, the corresponding arc, chord, sector, and segment also increase. Which increase at the same rate as the angle, and which do not?

2. Upon a given straight line as a chord describe a segment which shall contain a given angle.

Hints.—If AB is the given line, draw AC, making BAC equal the given angle. Then find the centre of the circle which has AB for a chord, and which AC touches at A. Then apply §§ **189** and **192**.

3. Construct a right triangle, having given the hypotenuse and a line in which the vertex of the right angle must lie.

4. In a given circle draw a chord parallel to a given straight line and having a given length.

5. Find the locus of all points from which tangents drawn to a given circle shall have a given length (§§ **73, 190**).

6. Given two circles; find the point from which tangents to the circles shall have the same length.

7. Draw a circle which shall touch two given parallel lines and pass through a given point lying between the lines. (Two solutions.)

8. About a given point describe a circle that shall touch a given circle. (The point may be either without or within the given circle.)

9. About each of two given points describe a circle so that the two circles shall touch each other, and one of them shall touch a given straight line.

Hint.—First construct the circle which touches the line, and then use the preceding exercise.

10. With a given radius describe a circle which shall touch a given circle and pass through a given point. (Discuss the problem.)

11. Construct a triangle, having given two sides and the radius of the circumscribed circle.

12. Construct a triangle, having given two angles and the radius of the inscribed circle.

13. Divide a given circle into four equal parts by describing concentric circles.

14. If the perimeter of a sector is equal to the circumference of the circle, find the angle at the centre.

15. If the area of a sector is equal to the square of the radius of the circle, find the angle at the centre.

16. If the perimeter of a sector is equal to twice the diameter of the circle, find the ratio of the sector to the circle.

CHAPTER X.

THE ELLIPSE.

CONTENTS.—I. The Properties of the Ellipse (§§ 219–221). II. Construction of Ellipses (§§ 222–224).

I.—The Properties of the Ellipse.

§ **219.** Among curved lines the *ellipse* ranks in importance next to the circle.

Fig. 210 shows how an ellipse may be described. Fasten a piece of paper upon a smooth surface, stick in two pins at any distance apart, and tie to the pins the ends of a string longer than the distance apart of the pins. Then describe the curve with a pencil, as shown in the figure, taking care to keep the string constantly stretched to its full length.

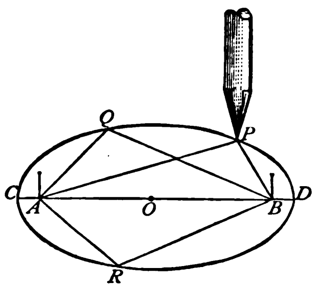

Fig. 210.

As the point of the pencil moves around the pins, its distance from either one of the pins is constantly changing; but it is clear that the sum of its distances from the two pins must always remain the same.

Thus, in *Fig. 210*, $AP + BP = AQ + BQ = AR + BR$, etc.

Definition.—*The* ELLIPSE *is a curve such that the sum of the distances of every point in it from two fixed points is the same.*

The two fixed points are called the *foci.*

The point O, half-way between the foci, is called the *centre.*

Define the ellipse regarded as the *locus* of a moving point.

The distance OA or OB from the centre to either of the foci, is called the *eccentricity* of the ellipse.

The less the eccentricity,—that is to say, the nearer the foci are to each other,—the wider the ellipse becomes in proportion to its length, and the more nearly it approaches to the form of the circle.

Describe several ellipses with their foci at different distances from each other.

What does the ellipse become if the foci coincide, or what comes to the same thing, if only one pin is used, to which both ends of the thread are fastened?

NOTE 1.—The ellipse is a more beautiful curve than the circle. Ellipses, or curves closely resembling ellipses, are much used in the arts; in cabinet work (tables, mirrors, etc.), in various designs and patterns, in the arches of bridges, and in architecture. It is to the skilful use of the ellipse and still more complex curves, that the superiority of Greek art over that of other nations is largely due. Astronomy presents us with examples of ellipses on the grandest scale. Our earth and all the planets of the Solar System revolve around the sun in elliptical paths in obedience to the law of gravitation, the sun being at one of the foci.

NOTE 2.—The ellipse is connected with the circle in a remarkable way. A section of a cylinder (page 4, *Fig. 3*) made by a plane parallel to the bases is (like the bases) a circle; but if the section is inclined to the bases, the boundary of the section is an ellipse; and the more inclined the section, the longer the ellipse in proportion to its breadth. All this is easily shown by making sections through a wooden cylinder.

If we look at a circle on paper with one eye placed directly in front of the centre of the circle, and then gradually turn the paper till its edge comes in front of the eye, we shall find that, as the paper is turned around, the circle will present the appearance of an ellipse which becomes narrower and narrower, and tends to pass finally into a straight line. In this way we pass, by imperceptible gradations, from the circle to the straight line.

§ **220.** In an ellipse $CDEF$ (*Fig. 211*), the line CD, which passes through the foci, and is limited by the ellipse, is called the *Major Axis;* and the line EF, which passes through the centre O perpendicular to the major axis, and is limited by the ellipse, is called the *Minor Axis.*

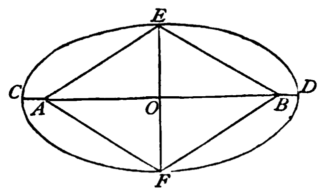

Fig. 211.

I. From the definition of the ellipse, it follows that $AC + BC = AD + BD$, or $2\,AC +$

$AB = 2\,BD + AB$; whence (Axiom III.) $2\,AC = 2\,BD$, and $AC = BD$; that is, —

The vertices of an ellipse are at the same distance from the foci.

II. Since $OA = OB$, and $AC = BD$; therefore (Axiom II.), $OA + AC = OB + BD$, or $OC = OD$; that is, —

The centre of an ellipse bisects the major axis.

III. Because (*Fig. 210*), $AP + BP = AC + BC$, and $AC = BD$; therefore, $AP + BP = BD + BC = CD$; that is, —

The sum of the distances of any point of the ellipse from the foci is equal to the major axis.

IV. Since $\triangle AOE \cong \triangle BOE$, therefore $AE = BE$; and since $AE + BE = CD$, therefore $2\,AE = CD$, and $AE = \frac{1}{2}CD = BE$; that is, —

The ends of the minor axis are equidistant from the foci, the distance being equal to half the major axis.

V. Since $\triangle AOE \cong \triangle AOF$, therefore $OE = OF$; that is, —

The centre of an ellipse bisects the minor axis.

Exercises. — **1.** To what are the three sides of the right triangle AOE (*Fig. 211*), respectively equal?

2. What kind of plane figure is $AEBF$ (*Fig. 211*)? Why?

3. Given the axes of an ellipse, find the positions of the foci.

4. Given the major axis and the positions of the foci, find the minor axis.

5. Semi-major axis $= 18^m$; semi-minor axis $= 12^m$. Find the eccentricity.

6. Semi-major axis $= 1.3^m$; eccentricity $= 0.34^m$. Find the semi-minor axis.

7. Semi-minor axis $= 2.9^m$; eccentricity $= 0.9^m$. Find the semi-major axis.

8. Major axis $= 58^{dm}$; minor axis $= 45^{dm}$. How far apart are the foci?

9. The path of the earth about the sun (at one focus) is an ellipse whose axes are 20,657,700 and 20,655,100 geographical miles. How can the least and the greatest distances of the earth from the sun be computed?

§ **221.** Area of an Ellipse. — If we describe circles about the axes of an ellipse as diameters (see *Fig. 213*), it is obvious that the area of the ellipse is less than that of the circle with the semi-major axis as radius, and greater than that of the circle with the

semi-minor axis as radius. Now it can be proved that the area of the ellipse is exactly equal to *that of the circle whose radius is a mean proportional between the semi-axes of the ellipse.*

If a and b are the semi-axes of an ellipse, r the radius of the circle equal in area to the ellipse, then $a : r = r : b$, or $r^2 = ab$. The area of this circle $= \pi r^2$ (§ **213**); therefore the area of the ellipse $= \pi r^2$; or, since $r^2 = ab$,

$$\textbf{Area of the ellipse} = \pi \mathbf{a\, b}. \qquad [12.]$$

That is to say, *the area of an ellipse is equal to the continued product of its semi-axes and the number π.*

Exercises.—Find the area of an ellipse, if,—

1. The semi-axes are 12^{dm} and 8^{dm}. Find the area.

2. The axes are 8.5^{m} and 4.5^{m}. Find the area.

3. The major axis $= 9^{m}$, the eccentricity $= 1.5^{m}$. Find the area.

4. The minor axis $= 2^{m}$, the eccentricity $= 1^{m}$. Find the area.

5. The major axis $= 5^{m}$, the focal distance $= 3^{m}$. Find the area.

6. The area $= 68^{qm}$, the major axis $= 12^{m}$. Find the minor axis and the eccentricity.

7. How much cloth will cover an elliptical table 2.4^{m} long, 1.3^{m} wide?

8. The famous amphitheatre at Verona, built by the Emperor Domitian, and which seated 24,000 spectators, has for its base an ellipse whose axes are 133^{m} and 105^{m}. Find the area of this ellipse.

9. The section of an arch has the shape of a semi-ellipse 16^{m} long, 4.8^{m} high. Find the area of the section.

10. The diameter of a circle $= 7^{m}$. Upon this diameter as major axis an ellipse is constructed half as large as the circle. Find its minor axis.

11. A pond in the shape of an ellipse, with the axes 60^{m} and 44^{m}, is changed to a circle with the radius 60^{m}. How much larger is it than before?

12. Find the radius of a circle equal in area to an ellipse whose axes are 13^{m} and 9^{m}.

13. Find the side of a square equal in area to an ellipse whose axes are 17^{m} and 10^{m}.

14. Find the area of the largest ellipse that can be made out of a board 2^{m} long and 0.6^{m} wide.

15. The axes of an ellipse are 4^{m} and 3^{m}. Upon these axes as diameters circles are described. Find the areas of all three figures.

II.—Construction of Ellipses.

§ **222.** **Problem.**—*To construct an ellipse, having given the major axis and the eccentricity.*

Let (*Fig. 212*) CD be the given major axis, O its middle point. Make $OA = OB =$ the given eccentricity; then A and B are the foci. Find the points E and F, such that the distance from each of them to A and to B is equal to the semi-major axis; E and F are the ends of the minor axis.

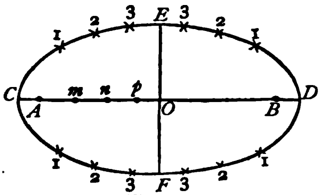

Fig. 212.

In order to find more points of the ellipse, assume any points m, n, p, etc., lying in the major axis between A and O. With a radius equal to Cm, describe arcs from each focus as centre, both above and below CD; and likewise do the same with a radius equal to Dm; the four points marked 1, where these arcs intersect, are four points of the ellipse. For, by this construction, the sum of the lines drawn from either of the points marked 1 to the two foci is equal to the major axis. (See § **220**, III.)

In like manner, by the use of n, the points marked 2, and by the use of p, those marked 3, are found.

By drawing (free-hand) through all these points a curved line, making the curvature as nearly uniform as possible, we obtain a curve which approaches the nearer to the exact ellipse required, the greater the number of points that are previously determined.

Exercises.—**1.** How can an ellipse be described by continuous motion (§ **219**). (Gardeners sometimes use this method to trace the outline of an elliptical flower-bed.)

2. Construct the ellipse whose major axis $= 80^{cm}$ and eccentricity $= 30^{cm}$, by finding sixteen points in the ellipse.

3. Construct the ellipse whose major axis $= 60^{cm}$ and eccentricity $= 16^{cm}$, finding fourteen points.

§ **223.** **Problem.** — *To construct an ellipse, having given the axes, by the aid of circles described about the axes as diameters.*

Let AB and CD (*Fig. 213*) be the given axes intersecting at O. Describe circles about AB and CD as diameters. Through O draw any radii OE, OF, etc. Through the points where these radii cut the inner circle draw lines ∥ AB, and through the points where the radii cut the outer circle draw lines ∥ CD. The intersection of each pair of these lines is a point in the required ellipse; so that nothing remains but to draw (free-hand) through all the points of intersection a smooth curve.

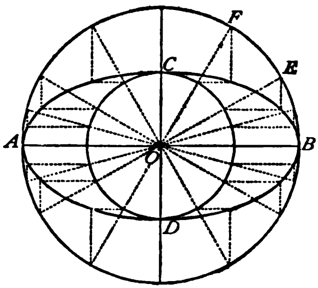

Fig. 213.

Exercises. — **1.** Construct an ellipse having for its axes 72^{cm} and 40^{cm}, and find the positions of the foci.

2. Construct an ellipse whose major axis is to the minor axis as 5 : 4, and find the positions of the foci.

§ **224.** **Problem.** — *To construct with arcs of circles a curve that resembles an ellipse.*

Draw a line AD (*Fig. 214*), and divide it into three equal parts, AB, BC, CD. With a radius equal to one of these parts describe about B and C as centres two circles, cutting each other in E and F. Through these points and the two centres draw the lines EG, EH, FI, FK. Then describe about E and F as centres, with a radius equal to EG, the arcs GH and IK. The curve $AGHDIK$ is the required curve.

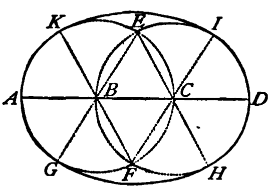

Fig. 214.

The elliptical curves employed in the arts are usually constructed in this or a similar way from arcs of circles. By compounding arcs of circles, in different ways, a great variety of curves may be constructed. Some examples are added below, in *Figs. 215–222.* Explain how each figure is constructed.

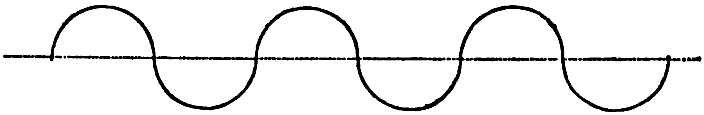

Fig. 215.

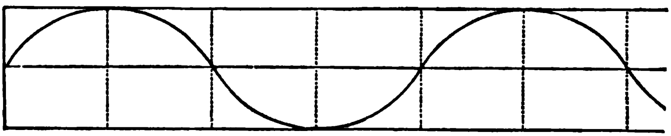

Fig. 216.

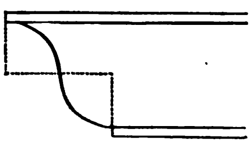

Fig. 217.

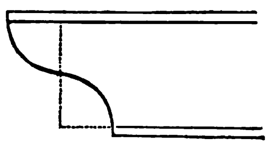

Fig. 218.

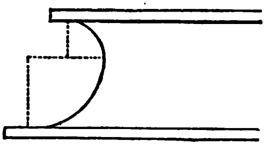

Fig. 219.

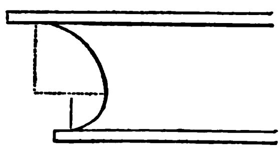

Fig. 220.

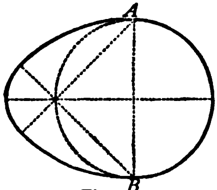

Fig. 221.

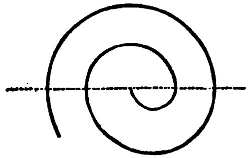

Fig. 222.

Exercises. — **1.** Construct an elliptical curve with arcs of circles.

2. Construct the curve shown in *Fig. 215*.

3. Construct the curve shown in *Fig. 216*.

4. Construct the curve shown in *Fig. 217*.

5. Construct the curve shown in *Fig. 218*.

6. Construct the curve shown in *Fig. 219*.

7. Construct the curve shown in *Fig. 220*.

8. Construct an *oval* like that in *Fig. 221*.

9. Construct a *spiral* like that in *Fig. 222*.

10. Find the area of an oval like that of *Fig. 221*, if the diameter $AB = 4^m$. Explain each step of the solution, stating the truth on which it depends.

SYNOPSIS OF CHAPTER X.

§ **219.** Definition of the *ellipse*, its *foci*, its *centre*, its *eccentricity*. Relation of the ellipse to the circle.

§ **220.** Definition of the *axes* of an ellipse. Relations between the ends of the axes, the foci, and the centre. Five general relations.

§ **221.** Area of an ellipse. Formula for the area.

§ **222.** Construction of an ellipse, given the minor axis and the eccentricity.

§ **223.** Construction of an ellipse with the aid of circles, given the axes of the ellipse.

§ **224.** Construction of a curve resembling an ellipse by means of arcs of circles. Other instances of the use of circular arcs in constructing curves.

CHAPTER XI.

PLANES.

CONTENTS.—I. Straight Lines in Space (§§ 225–227). II. A Plane (§§ 228–230). III. A Plane and a Straight Line (§§ 231–235). IV. Two Planes (§§ 236–239). V. Three Planes (§ 240). VI. Solid Angles (§ 241).

I.—Straight Lines in Space.

§ **225.** Thus far the figures studied (triangles, polygons, circles, etc.) are such as can be represented in their true shape on paper or the blackboard; they are figures which lie wholly in one plane.

We shall now proceed to consider some of the properties of figures which are not confined to one plane. Every actual body which we can see or handle is an example of such a figure.

We shall begin by examining the relations of lines and planes in space to one another.

NOTE 1.—In this part of Geometry the *first and chief* difficulty encountered by beginners lies in conceiving and holding firmly in the mind the true relative positions of the lines and planes under consideration. Without the power to do this the proofs of theorems amount to mere words; with it the proofs are easily mastered; in fact many of the theorems will seem self-evident. Accordingly, in the brief space here devoted to the subject, rigorous proofs will not always be given, but care will always be taken to set forth the truth so that it shall be completely realized in thought. The geometric imagination must be exercised and strengthened before the mind is prepared to study the subject with all the rigor of logic.

NOTE 2.—An additional difficulty arises from the fact that in representing magnitudes in different planes upon a single plane surface (that of paper or the blackboard) these magnitudes must, in general, be changed in size or shape or both. Lines and angles must be altered in size, plane figures in both size and shape. The rules for doing this correctly are based on the principles of Geometry, and constitute the Art of Perspective, of Isometric Drawing, etc. There is one general rule of Isometric Drawing which it is useful to bear in mind: *a square, not in the plane of the drawing, is represented in that plane by a rhombus, a rectangle by a parallelogram, and a circle by an ellipse.* Beginners should make a habit of comparing the figures in the book, and those which they draw themselves, with actual lines and planes placed in the proper relative positions, until they are able to recognize at once the true relative positions of the parts of a drawing, and to make similar drawings themselves. For lines, wires and stretched threads may be employed; for planes, paper, cardboard, a thin piece of board, glass, the floor and sides of the

room; for bodies, suitable models made of wood or cardboard (see *Figs. 1–6*) or constructed simply of stiff wire for the edges.

NOTE 3.—Some of the relations of lines and planes have been noticed in Chapter I. and in §§ **24, 25**. The learner should refresh his memory, if necessary, by reviewing them.

§ **226.** Two straight lines in space may have either of three relative positions, as follows:—

(*i.*) They may be parallel to each other.

(*ii.*) They may intersect each other.

(*iii.*) They may be neither parallel nor intersecting.

In the first two cases the lines must lie in the same plane (§ **29**). Give examples of both cases. If the lines intersect, they may be either perpendicular to each other or inclined to each other (§ **48**).

In the third case the two lines do not lie in the same plane. A line from north to south on the table, and a line from east to west on the floor, is an example of two such lines.

Let AB and CD (*Fig. 223*) represent two lines which are not parallel and do not intersect, and let E be the point of AB which is nearest to CD, and F the point of CD which is nearest to AB. Then *the line EF is the shortest distance between the lines AB and CD, and is a common perpendicular to both lines.*

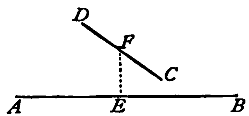

Fig. 223.

NOTE.—In *Fig. 223*, AB is to be regarded as in the plane of the paper, CD as nearly perpendicular to this plane.

§ **227.** Let ABC and DEF (*Fig. 224*) be two angles in space, and let $AB \parallel DE$, and $BC \parallel EF$. Conceive the angle ABC to move so that its vertex B keeps in the line BE, and AB and CB to remain parallel to their first positions. When B reaches E, it is evident that AB will coincide with DE, and CB with

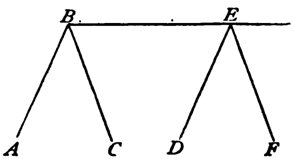

Fig. 224.

FE, and therefore the angle *ABC* will coincide with the angle *DEF*.

Therefore, the theorem of Plane Geometry (§ **58**) that *angles, whose sides are respectively parallel and directed the same way from the vertex, are equal,* also holds true of any two angles in space.

II.—A Plane.

§ **228**. The surface of still water, the surface of a table, the floor, or the blackboard, are examples of plane surfaces, or *planes*. Give another example. How can a plane be generated by motion? (See § **28**.)

How was a plane defined in § **28**? And what is the test of a plane surface?

The following is the more abstract but more precise definition commonly given by mathematicians: —

Definition. — A PLANE *is a surface such that the straight line which joins any two points in the surface lies wholly in the surface.*

A plane (like a straight line) may be conceived to extend indefinitely. The surface of the table may be extended in thought as far as we please, and in such a way that the surface which we imagine will be everywhere a part of the same plane as the actual surface which we see.

Exercise. — Give an instance of a vertical plane, a horizontal plane, an inclined plane.

§ **229**. When we conceive a plane as containing a given point or straight line, we are said *to pass the plane through the point or the line.*

Through *one* point an endless number of planes in all conceivable positions may be passed.

Through *two* points, also, we may pass as many different planes

as we please; for if we pass a plane through the line joining the points A and B (*Fig. 225*), and then rotate the plane about this line as an axis, it will assume different positions, in all of which it contains the line, and therefore the points A and B.

But if a *third* point C be given, through which the plane must also pass, it is evident that as we rotate the plane there is only one position in which it will contain all three points. Through three points not in the same straight line only one plane can be passed; and the same is obviously true of a straight line and a point not in the line.

It is likewise true of two intersecting lines and of two parallel lines; for if we pass a plane through one of the lines, and then turn the plane about this line, there is only one position in which it can contain the other line.

Therefore, *a plane is determined,—*

1. *By three points not in the same straight line.*
2. *By a straight line and a point not in the line.*
3. *By two intersecting lines.*
4. *By two parallel lines.*

§ **230**. Any two planes which are passed through a straight line AB (*Fig. 225*) have this line for their intersection; and it is quite clear that through a curved line only one plane at most can be passed.

The intersection of two planes is always a straight line.

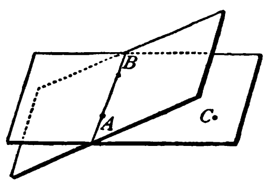

Fig. 225.

Exercises.—1. How many planes can intersect in the same straight line?

2. How many planes can be passed through a curve drawn on the blackboard?

3. Make (with a wire) a curve such that no plane can be passed through it.

III.—A Plane and a Straight Line.

§ **231.** A straight line may be either,—

(*i.*) Parallel to a plane.

(*ii.*) Perpendicular to a plane.

(*iii.*) Inclined to a plane.

An edge of the ceiling is parallel to the floor; a plumb-line is perpendicular to the surface of still water; a rafter of a common roof is inclined to the level ground. Give other examples.

In *Fig. 226*, the line *AB* is parallel to the plane *M*.[1] The line *PO* is perpendicular to *M*, and the lines *PC* and *PD* are inclined to *M*.

Exercises.—1. Draw a figure like *Fig. 226*, to illustrate the different positions which a straight line may have relative to a plane.

2. Hold a pencil, (*i.*) parallel, (*ii.*) perpendicular, (*iii.*) inclined, to the table.

§ **232. Definition.**—*A straight line and a plane are parallel, if they cannot meet however far extended.*

This condition will be fulfilled if the straight line *AB* (*Fig. 226*) is parallel to a straight line *CE* in the plane; for, since *AB* can never meet *CE*, and can never leave the plane determined by *AB* and *CE* (§ **229**, 4), it can never meet the plane *M* in which *CE* lies. Hence,—

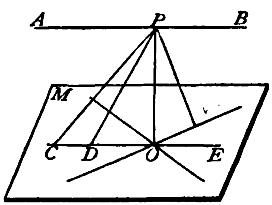

Fig. 226.

I. *A straight line is parallel to a plane if it is parallel to a straight line drawn in the plane.*

II. *If two lines are parallel, every plane passed through one of the lines will be parallel to the other line.*

What exception is there to this last statement?

[1] In *naming* a plane, a single letter, as *M* or *N*, is usually sufficient.

Exercises. — 1. How many lines parallel to a plane can be drawn through a point outside the plane?

2. If two lines are parallel to a plane, are they parallel to each other? Give illustrations.

§ **233.** A straight line, perpendicular or inclined to a plane, will (prolonged, if necessary) meet the plane in a point. This point is called the *foot* of the line.

Definition. — *A straight line is perpendicular to a plane, if it is perpendicular to every straight line that can be drawn through its foot in the plane.*

It is also said to be *normal* to the plane.

A line having this property can always be drawn from a point P (*Fig. 227*) to a plane M. For, among all the lines that can be drawn from P to meet the plane, there will be one, PO, shorter than any other. If we draw through its foot O any straight lines AB, CD, etc., PO will be the shortest line from P to each of these lines; therefore (§ **83**), perpendicular to each and all of them.

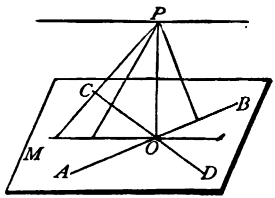

Fig. 227.

Hence, *a perpendicular measures the distance from a point to a plane.*

Through the intersection O (*Fig. 228*) of two lines, AB and CD, draw any line $OE \perp CD$, and revolve OE about CD, keeping $OE \perp CD$; there is one, and *only one*, position in which the line will also be $\perp AB$. Let OP be this position; then OP, being the only line through O perpendicular to both AB and CD, must be perpendicular to the plane passed through AB and CD. Therefore, —

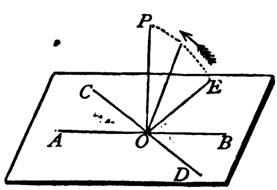

Fig. 228.

If a straight line is perpendicular to two straight lines drawn through a point of the line, it is perpendicular to the plane containing those two lines.

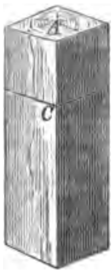

Fig. 229.

This proposition enables us to solve the problem: To make a plane perpendicular to a straight line. We have only to draw through any point of the line two perpendiculars, and then to pass a plane through these perpendiculars. Explain how a carpenter would proceed in order to saw squarely in two the beam of wood shown in *Fig. 229.*

Exercises. — 1. Give examples of a line perpendicular to a plane.

2. When a right triangle revolves about one of its legs, what is generated by the other leg?

3. At a point of a straight line in space, how many perpendiculars can be erected? From a point not in the line, how many perpendiculars can be let fall to the line?

4. Find the distance from a point P to a plane, if its distance from a point A in the plane $= a$, and the distance of this point from the foot of the perpendicular let fall from the first point to the plane $= b$. Solve for the case where $a = 17$, $b = 8$.

5. Prove that lines drawn from a point meeting a plane at equal distances from the foot of the perpendicular let fall from the point are equal; and that of two lines drawn to unequal distances from the foot of the perpendicular, the more remote is the greater.

6. Prove the converse of the first part of the last exercise.

7. Find the locus of points in a plane equidistant from points not in the plane.

8. Find the locus of points equidistant from two given points.

9. Find the locus of points equidistant from three given points.

§ **234.** Let AB and CD (*Fig. 230*) be normal to the plane M, and suppose AB to move along AC, keeping parallel to its first position. Then its position relative to the plane will not change, that is, it will remain normal to the plane; and when A coincides

with C, AB will coincide with CD. Therefore, $AB \parallel CD$. That is,—

Two lines that are normal to a plane are parallel to each other.

The converse is also true. State it.

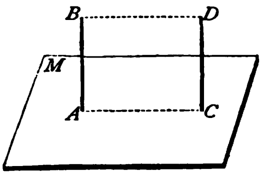

Fig. 230.

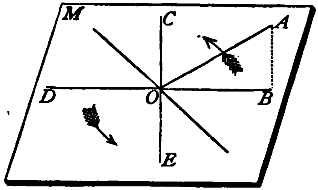

Fig. 231.

§ **235.** Lines neither parallel nor perpendicular to a plane are *inclined* to the plane.

Let AO (*Fig. 231*) be inclined to the plane M. Draw $AB \perp M$, and join OB; then OB is called the PROJECTION of the line AO upon the plane. When is the projection of a line equal in length to the line? less than the line? When is it a point?

The acute angle AOB, which AO makes with its projection BO, is the smallest angle which AO makes with any line drawn through its foot on the plane M. To see this more clearly, revolve BO about O in the plane M. The angle which the revolving line makes with AO will continually increase, becoming equal to a right angle AOC after one-fourth of a revolution, and attaining its greatest obtuse value AOD after half a revolution. During the remaining half of the revolution what changes does the angle undergo?

The angle which a line inclined to a plane makes with its projection in the plane, is called the *inclination* of the line to the plane.

NOTE.—What is said above may be illustrated by drawing lines through a point on a sheet of paper, and placing the end of the pencil held in an inclined position against this point. If in this position we let the pencil drop, the angle which it describes in falling is the inclination of the pencil to the table.

IV. — Two Planes.

§ **236**. Two planes may be either, —

(*i.*) Parallel to each other.

(*ii.*) Perpendicular to each other.

(*iii.*) Inclined to each other.

The floor and ceiling of the room are parallel planes; the floor and a side of the room are perpendicular to each other; the floor and the roof of the house are inclined to each other. Give other examples.

Exercises. — 1. In *Fig. 232*, which planes are parallel? perpendicular? inclined?

2. Draw two planes of each kind.

3. Hold two planes (books, thin sticks of wood, etc.) parallel; perpendular; inclined.

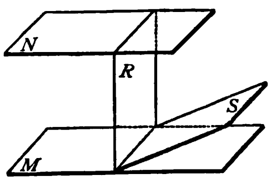

Fig. 232.

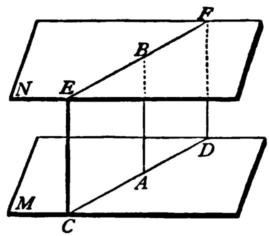

Fig. 233.

§ **237. Definition.** — *Two planes are parallel if they cannot meet however far extended.*

This condition is fulfilled if the planes M and N (*Fig. 233*) are both perpendicular to the same line AB. For if we pass a third plane through AB, it will intersect M in a line $CD \perp AB$ (why?), and N in a line $EF \perp AB$. Therefore, CD and EF, being in the same plane, are parallel (§ **57**), and can never meet. The same reasoning may be applied to all planes that can be passed through AB; therefore, the planes M and N can never meet.

Therefore,—

I. *Planes that have a common normal are parallel.*

The converse is also true. State it.

It is also evident from what precedes, that,—

II. *The intersections of two parallel planes by a third plane are parallel lines.*

If, also, $CE \parallel DF$ (*Fig. 233*), then $CDFE$ is a parallelogram (§ **101**), and $AC = BD$ (§ **102**); that is,—

III. *Parallel lines between parallel planes are equal.*

Among these equal parallel lines are the common normals (§ **234**), which measure the distance apart of the planes; therefore,—

IV. *Parallel planes are everywhere equally distant.*

§ **238. Definition.**—*Two planes are perpendicular to each other if a normal to either plane, erected at any point of their intersection, lies in the other plane.*

The angle made by the normals to the two planes, erected at the same point of the intersection, is a right angle (why?).

Let the planes M and N (*Fig. 234*) intersect in the line AB, and let CD be normal to M and lie in N.

Draw DE normal to N; then CDE is a right angle (why?); therefore DE *must lie in* M, the plane containing all lines perpendicular to CD at D.

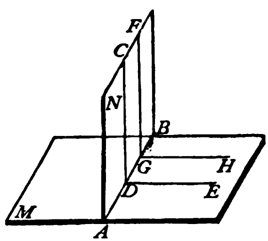

Fig. 234.

Further, let G be any other point of the intersection, and draw GF normal to M, and GH normal to N. Then $GF \parallel DC$ (§ **234**); and since DC lies in the plane N, and G also lies in N, therefore GF *must lie in* N. For like reasons, GH *lies in* M.

Comparing these results with the definition, we see that,—

I. *Two planes must be perpendicular, if a single normal to one of the planes erected at a point of the intersection lies in the other plane.*

II. *Every plane passed through a normal to a plane is perpendicular to that plane.*

Exercises. — **1.** How can a plane perpendicular to a given plane be passed through a given point?

2. How can a plane perpendicular to a given plane be passed through a straight line, (*i.*) when the line lies in the given plane? (*ii.*) when it is inclined to the given plane? (*iii.*) when it is parallel to the given plane?

§ **239.** Two *inclined* planes M and N (*Fig. 235*) intersect each other and form what is called a DIHEDRAL ANGLE.[1] This angle may be conceived as generated by the revolution of one of the planes M about the line AB until it comes into the position of the other plane N. The planes M and N are called the *faces* of the angle; the intersection AB is called the *edge* of the angle.

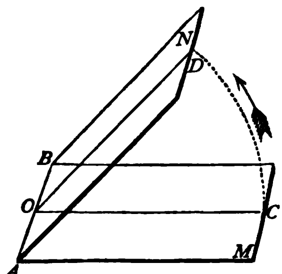

Fig. 235.

A dihedral angle is always measured by the angle of two lines. If OC is $\perp$ AB, during the revolution of M about AB, OC describes the angle COD, which has the same value wherever the point O is situated in the line AB (§ **227**). This angle may therefore be used to measure the dihedral angle made by the planes. That is, —

A dihedral angle is measured by the angle made by two lines drawn through any point of the edge, perpendicular to the edge, one line in one plane, the other in the other plane.

Exercises. — **1.** Illustrate, by the leaves and edges of a book, dihedral angles, and how they are measured.

2. If the angle of the lines which measure a dihedral angle is 90°, what position do the planes have?

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[1] From Greek words meaning *two-faced*.

V.—*Three Planes.*

§ **240.** Three planes may have five different relative positions, shown in *Figs. 236–238.*

(*i.*) In *Fig. 236*, the planes are parallel to one another.

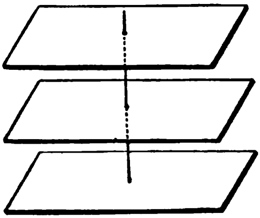

Fig. 236.

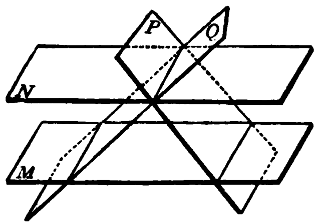

Fig. 237.

(*ii.*) In *Fig. 237*, the two planes *M* and *N* are parallel, and the third plane *P* (or *Q*) intersects *M* and *N* in parallel lines. (See § **237**.)

(*iii.*) In *Fig. 237*, the planes *N*, *P*, and *Q* have a common line of intersection.

(*iv.*) In *Fig. 237*, *M*, *P*, and *Q* intersect in three lines, which are parallel to one another.

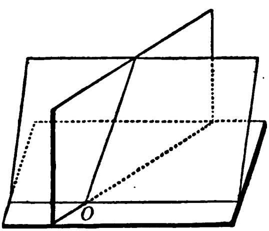

Fig. 238.

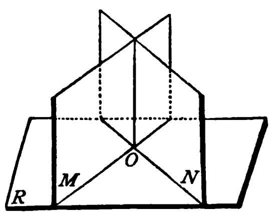

Fig. 239.

(*v.*) In *Fig. 238* there are also three lines of intersection. These lines meet in a common point *O*; so that *O* is a point common to all the planes.

If two of the planes M and N (*Fig. 239*) are both perpendicular to the third plane R, it follows, from § **238**, that their intersection must be a normal to the plane R; for the normal to R, erected at the point O, must lie in both M and N.

Exercise. — Give an example of three parallel planes; of two planes, both perpendicular to a third plane.

VI. — *Solid Angles.*

§ **241**. When three or more planes meet in one point O (*Fig. 241*) they are said to form at that point a SOLID ANGLE. The planes are called the *faces*, their lines of intersection the *edges*, and the angles at O, made by the two edges of each plane, the *plane* angles of the solid angle.

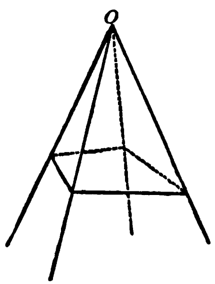

Fig. 240.

If there are only three planes, the angle is usually called a *trihedral* angle.

In order to form a trihedral angle from three angles in one plane, $AOB = a$, $AOC = b$, $BOD = c$ (*Fig. 241*), of which we will suppose a to be the greatest, turn the plane AOC about AO, and the plane AOD about BO, till they meet, if possible, and the lines CO and DO coincide.

If $b + c = a$, CO and DO will just coincide when turned through half a revolution, so that both again lie in the plane of the paper; if $b + c < a$, CO and DO cannot be made to coincide. In both cases no trihedral angle will be formed. But if $b + c > a$, CO and DO will coincide in a line OE, either in front of the plane of the paper (*Fig. 242*), or behind it (*Fig. 243*), according as the motior is *forward* or *backward;* and a trihedral angle is formed. And, since a is supposed the greatest of the three angles, it follows that $a + b > c$, and that $a + c > b$. Hence, *in every trihedral ang the sum of any two of the plane angles is greater than the third.*

If the motion is forwards, the trihedral angle shown in *Fig. 242* is formed; if backwards, the trihedral angle shown in *Fig. 243*. These two angles are composed of equal plane angles, taken in

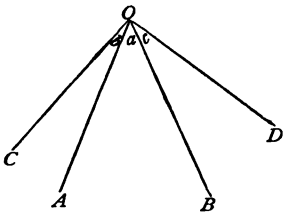

Fig. 241.

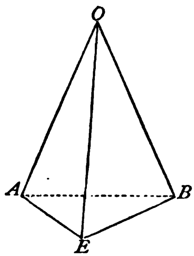

Fig. 242.

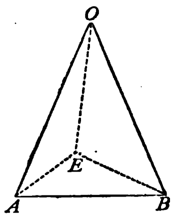

Fig. 243.

order; yet they cannot be made to coincide, because their equal parts follow in *reverse* order, — in one, from left to right; in the other, from right to left.

They stand to each other in the same relation as an object to its image in a mirror, or as the right hand to the left. Two such solid angles are said to be *symmetrical.*

Two solid angles are *equal,* if they are composed of equal plane angles taken in the *same* order. We may conceive two equal solid angles placed one upon the other so as to coincide in all their parts.

Whatever be the number of plane angles which compose a solid angle, *their sum, estimated in degrees, must be less than 360°;* if it were equal to 360°, the plane angles would necessarily lie in one plane, and would not form a solid angle.

Exercises. — 1. Give examples of solid angles.

2. In a solid angle, how is the angle which two faces make with each other measured?

3. Draw a solid angle having three faces; four faces; six faces.

REVIEW OF CHAPTER XI.

I. — Straight Lines in Space.

§ **225.** Nature of Solid Geometry.
§ **226.** Two straight lines in space.
§ **227.** Two angles in space.

II. — A Plane.

§ **228.** Definition of a plane.
§ **229.** Determination of a plane.
§ **230.** Intersection of two planes.

III. — A Straight Line and a Plane.

§ **231.** Positions of a straight line with respect to a plane.
§ **232.** The straight line parallel to a plane.
§ **233.** The straight line perpendicular to a plane.
§ **234.** Two normals to a plane.
§ **235.** The straight line inclined to a plane.

IV. — Two Planes.

§ **236.** Relative positions of two planes.
§ **237.** Two parallel planes.
§ **238.** Two perpendicular planes.
§ **239.** Two inclined planes.

V. — Three Planes.

§ **240.** Relative positions of three planes.

VI. — Solid Angles.

§ **241.** Solid angles. Symmetrical solid angles. Equal solid angles.

CHAPTER XII.

GEOMETRICAL BODIES.

I.—Polyhedrons.

§ **242.** A limited portion of space regarded as possessing solely form and magnitude is called a GEOMETRICAL BODY, or a SOLID (see § **7**).

In this sense a room is a body as much as a stone or piece of wood. Speaking exactly, it is the space occupied by the air, etc., in the room, and limited by the floor, sides, and ceiling of the room.

A geometrical body bounded on all sides by planes is called a POLYHEDRON.

Give examples of polyhedrons.

The bounding planes are called the *faces* of the polyhedron, their intersections are called the *edges*, and the intersections of the edges are called the *corners*.

The faces that meet at each corner form a solid angle, composed of as many plane angles as there are faces.

A polyhedron must have at least *four* faces; for it takes at least three planes to form a solid angle, and a fourth plane is required to close the opening between the three planes.

A polyhedron with four faces is called a TETRAHEDRON; with five faces, a PENTAHEDRON; with six faces, an HEXAHEDRON; with eight faces, an OCTAHEDRON; with ten faces, a DECAHEDRON; with twelve faces, a DODECAHEDRON; with twenty faces, an ICOSAHEDRON.

A very simple relation exists between the faces, corners, and edges of a polyhedron.

If we examine the polyhedrons in *Fig. 2*, page 4, we find that, —

No. I. has	5 faces,	6 corners, and	9 edges.
II. has	6 "	8 "	" 12 "
III. has	8 "	12 "	" 18 "
IV. has	5 "	5 "	" 8 "

In each of these cases, the number of faces + number of corners = number of edges + 2.

This relation is perfectly general, and is called Euler's Theorem.[1] Let F denote the number of faces, C the number of corners, E the number of edges; then we may state Euler's Theorem as follows: —

In every polyhedron, $\mathbf{F + C = E + 2.}$ [13.]

§ **243.** A polyhedron whose faces are equal regular polygons, and whose solid angles are all equal, is called REGULAR.

It is easy to show that only five regular polyhedrons are possible.

We know already that at least three faces are necessary to form a solid angle; also, that the sum of the plane angles at each corner must be less than four right angles (§ **241**).

The angle of the equilateral triangle $= 60°$, and $3 \times 60° = 180°$; therefore, three such triangles may form a solid angle. Likewise, four such triangles may be used, since $4 \times 60° = 240°$; also five such triangles, since $5 \times 60° = 300°$. But $6 \times 60° = 360°$, so that six, or more than six, equilateral triangles cannot form a solid angle.

Therefore, only three regular polyhedrons are possible whose faces are equilateral triangles.

The angle of a square $= 90°$. Since $3 \times 90° = 270°$, while $4 \times 90° = 360°$, three is the only number of squares that can form a solid angle.

The angle of a regular pentagon $= 108°$. Since $3 \times 108° = 324°$, while $4 \times 108° = 432°$, only three regular pentagons can be used to form a solid angle.

[1] Euler was a famous German mathematician, born 1707, died 1783.

The angle of a regular hexagon = 120°. Since 3 × 120° = 360°, three hexagons placed together with a corner common would lie in one plane and could not form a solid angle. Therefore, no solid angle can be formed with regular hexagons; still less with regular polygons having more than six sides.

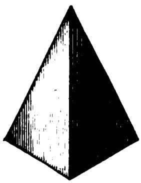

Fig. 244.

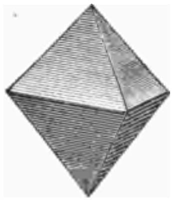

Fig. 245.

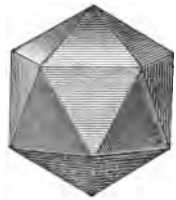

Fig. 246.

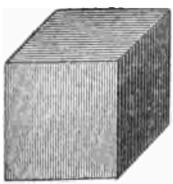

Fig. 247.

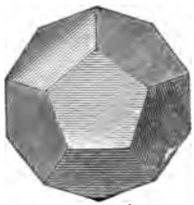

Fig. 248.

Therefore, only five regular polyhedrons are possible. They are,—

(1) The regular Tetrahedron (*Fig. 244*), bounded by four equilateral triangles.

(2) The regular Octahedron (*Fig. 245*), bounded by eight equilateral triangles.

(3) The regular Icosahedron (*Fig. 246*), bounded by twenty equilateral triangles.

(4) The regular Hexahedron (*Fig. 247*), bounded by six squares.

(5) The regular Dodecahedron (*Fig. 248*), bounded by twelve regular pentagons.

To DEVELOP[1] the surface of a solid is to draw in one plane the various parts of the surface, so that when folded up they shall form the solid.

The surface of the solid, when drawn in one plane, is called the DEVELOPMENT of the surface.

The surfaces of many solids cannot be developed; but when this can be done, it is easy to construct by this means *models* of the solids.

The surfaces of the regular polyhedrons are all developable, and their plane models are shown in *Figs. 249–253*.

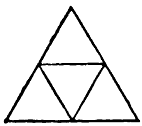

Fig. 249.

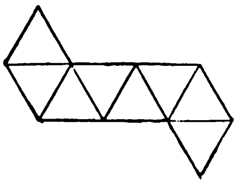

Fig. 250.

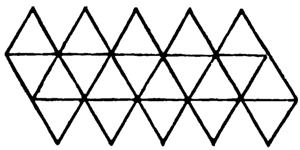

Fig. 251.

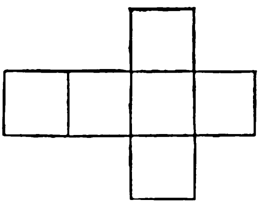

Fig. 252.

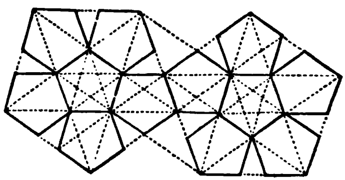

Fig. 253.

To make models of these solids, draw on cardboard the diagrams shown in these figures, cut them out of the cardboard, and then cut half through the edges of the adjacent polygons. The figures will then readily fold up into the shape of the solids, and can be retained in shape by means of glue or paste.

[1] Literally, *unroll.*

Exercises. — **1.** Give examples of polyhedrons.

2. How many edges have each of the regular polyhedrons?

3. How many corners have each of the regular polyhedrons?

4. Verify Euler's theorem for each of the regular polyhedrons?

5. Construct models of the regular polyhedrons.

NOTE. — In making the model of the regular dodecahedron, the diagonals, shown dotted in *Fig. 253*, may be used to advantage.

II. — The Prism, Cylinder, Pyramid, and Cone.

§ **244.** If the polygon $ABCDE$ (*Fig. 254*) move along the line AF, keeping parallel to its first position, its sides will describe parallelograms, and the polygon will generate a solid called a *Prism.*

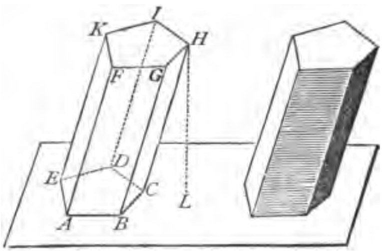

Fig. 254.

Definition. — A PRISM *is a polyhedron bounded by parallelograms and by two equal and parallel polygons.*

The parallelograms are called the *lateral faces*, their lines of intersection the *lateral edges*, and the two polygons the *bases* of the prism.

The *height* of the prism is the distance (HL) between the bases.

A prism is called *triangular*, *quadrangular*, etc., according as the base is a triangle, quadrilateral, etc.

A prism is *oblique* if the lateral edges are inclined to the bases (*Fig. 255*); *right*, if they are normal to the bases (*Fig. 256*); and *regular*, if it is a right prism with a regular polygon for its base.

A *parallelopiped* is a prism whose bases are parallelograms. It is, therefore, a solid bounded by six parallelograms.

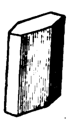
Fig. 255.

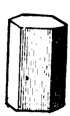
Fig. 256.

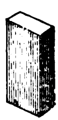
Fig. 257.

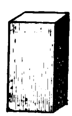
Fig. 258.

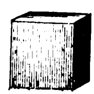
Fig. 259.

What is an *oblique* parallelopiped (*Fig. 257*)? A *right* parallelopiped (*Fig. 258*)?

A *rectangular* parallelopiped is a right parallelopiped, all of whose faces are rectangles.

A *cube* is a right parallelopiped, all of whose faces are squares (*Fig. 259*).

Exercises. — **1.** Can you give an example of a prism? What kind of a prism is it?

2. In *Fig. 254*, point out lateral faces, lateral edges, the bases, the height.

3. Name a prism with a base of five sides, six sides, eight sides.

4. What are the lateral faces of a right prism? Why is a lateral edge equal to the height of the prism?

5. How many faces, edges, corners, in a prism with a base of three sides, four sides, six sides, eight sides, twenty sides? Apply Euler's formula to each one of these cases.

6. Draw the following: —

(*i.*) Oblique triangular prism.	(*v.*) Oblique parallelopiped.
(*ii.*) Right pentagonal prism.	(*vi.*) Right parallelopiped.
(*iii.*) Right hexagonal prism.	(*vii.*) Rectangular parallelopiped.
(*iv.*) Regular hexagonal prism.	(*viii.*) Cube.

§ **245.** If a circle (*Fig. 260*) move in the direction normal to its own plane, keeping always parallel to its first position, it generates a solid called the *right circular Cylinder.* The same solid is also generated by a rectangle $ABCD$ (*Fig. 257*), which revolves about one side CD.

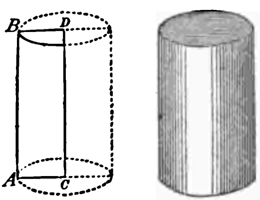

Fig. 260.

Definition. — *A right circular* CYLINDER *is the solid produced by the revolution of a rectangle about one of its sides.*

NOTE. — In future, by the word "cylinder" the right circular cylinder will be meant.

A cylinder is bounded by a curved surface called the *convex* surface, and by two equal and parallel circles called the *bases.* The line joining the centres of the bases is called the *axis;* its length is equal to the *height* of the cylinder.

§ **246.** If the polygon $ABCDE$ (*Fig. 261*) move along the line AO, keeping parallel to its first position, but diminishing in size at a uniform rate, *without altering in shape*, until it is reduced to a point O, its sides will describe triangles, and the polygon will generate a solid called a *Pyramid.*

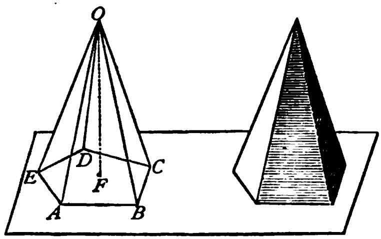

Fig. 261.

Definition.—A PYRAMID *is a polyhedron bounded by triangles that have a common vertex, and by a polygon whose sides are the bases of the triangles.*

The triangles are called the *lateral faces;* their lines of intersection, the *lateral edges;* their common vertex, the *vertex* of the pyramid; and the polygon, the *base* of the pyramid.

The *height* of the pyramid is the distance (OF) from the vertex to the base.

A pyramid is *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

A *regular* pyramid is a pyramid which has a regular polygon for its base, and in which the perpendicular from the vertex passes through the centre of the base (*Fig. 262*). The distance OM from the vertex to a side of the base is called the *slant height.*

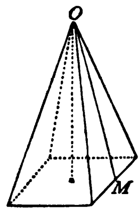

Fig. 262.

Exercises.—**1.** In *Fig. 261*, point out lateral faces, lateral edges, the base, the vertex, the height.

2. Name a pyramid with a base of five sides; 6 sides; 8 sides.

3. What kind of a pyramid is the tetrahedron (*Fig. 244*)?

4. What kind of triangles are the faces of a regular pyramid?

5. Draw (*i.*) a triangular pyramid; (*ii.*) a regular hexagonal pyramid.

§ **247.** If a circle (*Fig. 263*) move in a direction normal to its own plane, keeping parallel to its first position, but diminishing in size at a uniform rate till it is reduced to a point O, it will generate a solid called a *right circular Cone*.

The same solid is also generated by a right triangle AOB, which revolves about the leg OB.

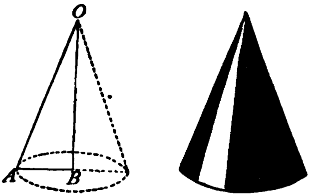

Fig. 263.

Definition. — *A right circular* CONE *is the solid produced by the revolution of a right triangle about one of its legs.*

NOTE. — In future, by the word "cone" a right circular cone will be meant.

A cone is bounded by a curved surface called its *convex* surface, and by a circle called its *base*. The point O is the *vertex* of the cone, and the line OB joining the vertex to the centre of the base is the *axis;* its length is equal to the *height* of the cone.

The length of the line AO (*Fig. 263*) is called the *slant height.*

§ **248.** From the mode of generation of the four bodies now described, — Prism, Cylinder, Pyramid, Cone, — it follows that a section made by a plane parallel to the base is, —

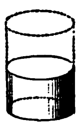

Fig. 264.

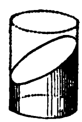

Fig. 265.

Fig. 266.

In a prism, a polygon equal to the base.

In a cylinder, a circle equal to the base (*Fig. 264*).

In a pyramid, a polygon similar to the base (*Fig. 271*).

In a cone, a circle smaller than the base (*Fig. 272*).

The section of a cylinder made by a plane inclined to the base

is an ellipse (*Fig. 265*). Such sections are made when two cylinders intersect each other, as in the case of two pipes meeting at right angles (*Fig. 266*).

In a cone, a section inclined to the base cuts the surface in an *ellipse* (*Fig. 268*), if it does not meet the base; in a *parabola* (*Fig. 269*), if it meets the base and is parallel to the side of the cone; in a *hyperbola* (*Fig. 270*), if it meets the base and is not parallel to the side of the cone.

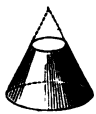

Fig. 267.

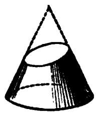

Fig. 268.

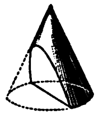

Fig. 269.

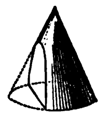

Fig. 270.

The four sections shown in *Figs. 267–270* — namely, circle, ellipse, parabola, hyperbola — are sometimes called the *conic* sections.

NOTE. — These sections may be shown to the eye by holding a cone partly under water, at first with the axis vertical, then with the axis more and more inclined. If the water is colored red, the effect is better seen.

Exercises. — What is the section made by a plane passing through, —

1. Two lateral edges of a prism?
2. Two lateral edges of a pyramid?
3. The axis of a cylinder?
4. The axis of a cone?

§ **249.** A plane parallel to the base of a pyramid, or of a cone, divides the body into a smaller pyramid or cone, and a body called the *frustum* of the pyramid or cone (*Figs. 271* and *272*). The base and the section parallel to the base are called the *lower* and *upper* bases of the frustum; their distance from each other is the *height* of the frustum.

If we know the height PQ (*Fig. 271*) of a frustum of a pyramid, and two homologous sides AB and EF of the bases, we can find the height OP of the entire pyramid, as follows: —

Through OP and the edge OA pass a plane intersecting the

bases of the frustum in the parallel lines AP and EQ (why parallel?), and draw $EM \parallel OB$. By § **159**, —

$$\frac{OQ}{PQ}=\frac{OE}{EA}, \text{ and } \frac{OE}{EA}=\frac{EF}{AM}.$$

Therefore, —

$$\frac{OQ}{PQ}=\frac{EF}{AM}, \text{ and } OQ=\frac{PQ\times EF}{AM}=\frac{PQ\times EF}{AB-EF}.$$

And $$OP=OQ+QP.$$

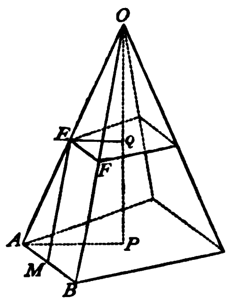

Fig. 271.

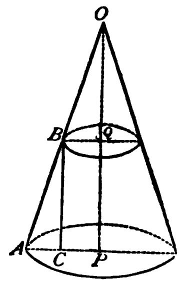

Fig. 272.

Exercises. — **1.** What are the faces of the frustum of a pyramid?

2. If (*Fig. 271*) $PQ=5^m$, $AB=4^m$, $EF=2^m$, find OQ and OP.

3. Given the height PQ (*Fig. 272*) of the frustum of a cone, and the radii of the bases; find the height of the cone cut off, and also the height of the entire cone.

Hint. — Draw $BC \parallel OP$, and use the similar triangles OBQ and ABC.

4. If (*Fig. 272*) $PQ=4^m$, $AP=3^m$, $BQ=1^m$, find OQ and OP.

§ **250**. The surfaces of the prism, the pyramid, the cylinder, and the cone are all developable.

Fig. 273 shows the development of a regular hexagonal prism; *Fig. 274* that of a regular pyramid whose base is a square.

The auxiliary lines in the two figures serve to illustrate how they are to be constructed. *Fig. 275* is the development of a regular hexagonal pyramid. How is it constructed?

The development of the convex surface of a cylinder is a rectangle; for we can roll up a rectangular sheet of paper, for example, into the shape of a cylindrical surface; and conversely, a cylindrical surface can be unrolled into a rectangle (*Fig. 276*).

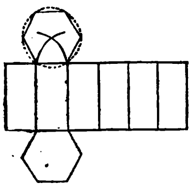

Fig. 273.

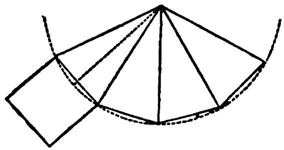

Fig. 274.

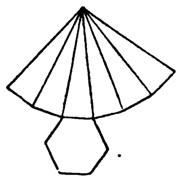

Fig. 275.

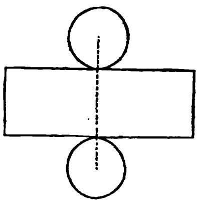

Fig. 276.

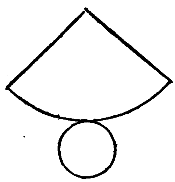

Fig. 277.

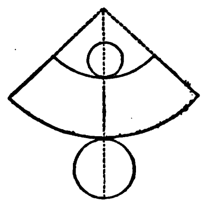

Fig. 278.

The development of the convex surface of a cone is a circular sector whose arc is the circumference of the base of the cone, and whose radius is the slant height of the cone (*Fig. 278*).

Fig. 276 shows the development of the entire surface of a cylinder; *Fig. 277*, that of the entire surface of a cone; *Fig. 277*, that of the entire surface of the frustum of a cone.

Exercises. — Make models of the following solids: —

1. A regular triangular prism.
2. A regular hexagonal prism.
3. A right prism, not regular.
4. A rectangular parallelopiped.
5. A regular hexagonal pyramid.
6. A cube.
7. A cylinder.
8. A cone.
9. A frustum of a pyramid.
10. A frustum of a cone.

III. — The Sphere.

§ **251. Definition.** — A SPHERE *is the solid produced by the revolution of a semicircle (or circle) about the diameter (Fig. 279).*

NOTE. — Spin a piece of money rapidly on the table, and it will present the appearance of a sphere.

The semi-circumference PMQ describes the surface of the sphere; and it is clear that this surface must be everywhere equally distant from O, the centre of the semicircle This point O is called the *centre* of the sphere.

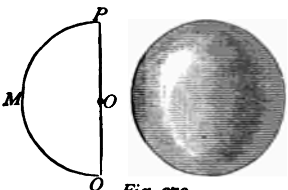

Fig. 279.

Therefore, a sphere may also be defined as a *solid bounded by a surface which is everywhere equally distant from a point called the centre.*

What is a *radius* of a sphere? a *diameter?*

An orange may be taken as a good example of a sphere. If we cut an orange into a series of *parallel* slices, the planes of division between the slices are circles differing in size, the largest being that which passes through the centre of the orange.

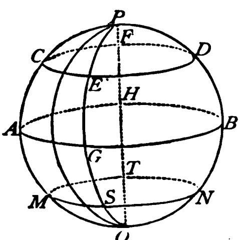

Fig. 280.

Every section of a sphere made by a plane is a circle.

If the section passes through the centre of the sphere, it is called a *great circle;* in all other cases, it is called a *small circle.*

The equal parts into which a great circle divides the sphere are called *Hemispheres.*

In *Fig. 280,* the sections $PAQB$ and $AGBH$ are great circles; the sections $CEDF$ and $MSNT$ are small circles.

All great circles of a sphere are equal; they have the same centre, the same radius, the same diameter, as the sphere itself.

All great circles divide each other into two equal parts; every great circle divides the sphere and its surface into two equal parts.

Small circles are the smaller the further they are from the centre of the sphere.

Every small circle divides the sphere and its surface into two unequal parts.

The centre of any circle of the sphere must lie in the diameter of the sphere which is normal to the plane of the circle; the two points where this diameter meets the surface of the sphere are called the *Poles* of the circle. In *Fig. 280*, the points P and Q are the poles of what circles?

On the earth (or terrestrial sphere), *parallels of latitude* are small circles, the *equator* is a great circle, and the north and south poles are the poles of all these circles.

All points in the circumference of a circle (great or small) are equidistant from either one of its poles, whether this distance be measured in a straight line or along the arc of a great circle passing through the point and the pole. Since the latter way of measuring distances is much more convenient on a sphere, it is always employed. Thus, in *Fig. 280*, the distance from the point C to the pole P is the length of the arc PC.

Of all lines that can be drawn *on the surface* of a sphere from one point to another, the arc of the great circle passing through the two points is the shortest. In this respect, the arc of a great circle is to the surface of a sphere what the straight line is to the plane.

By means of arcs of great circles (never small circles), *spherical triangles* and *spherical polygons* may be drawn on the surface of a sphere.

The portion of the surface of a sphere comprised between the circumference of two parallel circles is called a *Zone*.[1] The dis-

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[1] From a Greek word, meaning *belt* or *girdle*.

tance between the planes of the circles is the *height* of the zone, and the circumferences of the circles are its *bases*. If one of the planes touches the sphere without cutting it (or is a *tangent* plane), the zone is called a *zone of one base*. Point out zones in *Fig. 280*.

Exercises. — 1. Give examples of spheres (globes, balls).

2. Draw a figure of a sphere with lines in it to illustrate the definitions of this section.

3. How will a sphere be divided by three great circles mutually perpendicular to each other?

4. In what zones is the surface of the earth divided? Which are zones of one base?

§ **252**. An orange, after the peel is removed, is easily divided into wedge-shaped slices, separated by natural planes of division. These planes are planes of great circles intersecting in a common diameter. In the fruit, the number of the planes is limited; but, in any sphere, we may imagine as many such planes as we please, all passed through a common diameter, and each cutting the surface of the sphere in the circumference of a great circle.

Fig. 281 shows several such planes passing through the diameter PQ. They intersect the surface in the circumferences, the front parts of which are the semi-circumferences PAQ, PBQ, etc.

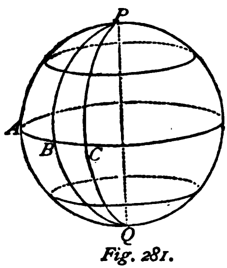

Fig. 281.

These semi-circumferences are called *Meridians*, and the portion of the surface of the sphere comprised between two meridians is called a *Lune*. Point out lunes in *Fig. 281*.

Terrestrial Meridians are the circumferences of great circles on the earth's surface made by planes passing through the earth's axis.

The surface of a sphere is not developable, but a close approxi-

mation to it can be constructed by drawing a series of equal lunes in one plane, as follows: —

Divide a line *AB* (*Fig. 282*), which must be made equal to $3\frac{1}{7}$ times the diameter of the proposed sphere, into twelve equal parts, and add nine of these parts to each end of the line. Then, with a

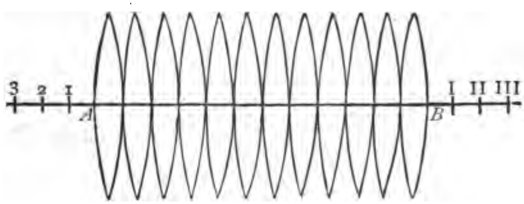

Fig. 282.

radius equal to ten of these parts, describe arcs from the points marked 1, 2, 3, etc.; and, likewise, arcs from the points marked I., II., III., etc., as shown in the figure. These arcs will inclose lunes which, cut out and properly folded up, will very nearly form a complete spherical surface.

It is by a process of this sort that the surfaces of balloons, base-balls, globes, etc., are made.

Exercises. — 1. Describe how a hemisphere would be generated by motion.

2. Make a model of a sphere.

REVIEW OF CHAPTER XII.

I. — Polyhedrons.

§ **242.** Definition of the Polyhedron. Specific names of polyhedrons. Euler's theorem.

§ **243.** The five regular polyhedrons. Development of their surfaces. How models of them are made.

II. — The Prism, Cylinder, Pyramid, and Cone.

§ **244.** Definition of the Prism. Different kinds of prisms. The parallelopiped.

§ **245.** Definition of the Cylinder.

§ **246.** Definition of the Pyramid. Different kinds of pyramids.

§ **247.** Definition of the Cone.

§ **248.** Sections of the prism, cylinder, pyramid, and cone.

§ **249.** Frustum of a pyramid or a cone.

§ **250.** Development of the surfaces of these four solids. How models of them are made.

III. — The Sphere.

§ **251.** The Sphere defined. Section of a sphere made by a plane. Great and small circles. Spherical triangles and polygons. Zones.

§ **252.** Meridian lines. Lunes. How a model of a sphere may be made.

CHAPTER XIII.

SURFACES AND VOLUMES OF BODIES.

I.—Surfaces of Bodies.

§ **253.** THE PRISM. The lateral surface consists of parallelograms; the bases are polygons. Therefore,—

I. *Lateral surface = sum of the areas of the parallelograms of which it is composed.*

II. *Total surface = lateral surface + areas of the bases.*

Exercises. — Find the total surface of a cube if,—

1. The edge $= 8^{m}$.

2. The edge $= 3.5^{m}$.

3. The edge $= 4^{ft}$.

4. The edge $= 7\frac{3}{4}^{in}$.

5. If the edge of a cube contain a unit of length, how many units of area are there in the entire surface? *Ans.* $6a^2$.

6. The total surface of a cube $= 600^{qm}$. Find the length of the edge.

7. If the total surface of a cube contains S units of area, how many units of length are there in one edge? *Ans.* $\sqrt{S \div 6}$.

8. Find the total surface of a rectangular parallelopiped if the length $= 15^{m}$, the breadth $= 9^{m}$, the height $= 6^{m}$.

9. Find the total surface of a rectangular parallelopiped if the length $= a$, breadth $= b$, height $= c$. *Ans.* $2ab + 2ac + 2bc$.

10. How many bricks, 8^{in} by $3\frac{1}{2}^{in}$, are required to line the bottom and sides of a reservoir 60^{ft} long, 24^{ft} wide, 12^{ft} deep?

Find the total surface of the following regular prisms:—

11. Triangular prism, height $= 76^{cm}$, side of base $= 40^{cm}$.

12. Quadrangular prism, height $= 6^{ft}$, side of base $= 27^{in}$.

13. Hexagonal prism, height $= 2.46^{m}$, side of base $= 84^{cm}$.

Hints.—Find less radius of base by use of § **202**, Case 2, and § **142**, and area of base by § **133**.

14. How would Exercise 13 be solved if the less (or the greater) radius of the base were given instead of a side of the base?

15. The total surface of a regular hexagonal prism $= 822^{qm}$, a side of the base $= 10^{m}$. Find the height (see Table, p. 221).

16. In a regular hexagonal prism, the height $= h$, a side of the base $= a$. Find the total surface. *Ans.* $6ah + 3a^2\sqrt{3}$.

§ **254.** THE CYLINDER. The development of the convex surface is a rectangle (*Fig. 276*) whose base is equal to the circumference of the base of the cylinder, and whose altitude is equal to the height of the cylinder. The bases of the cylinder are equal circles. Therefore (§ **128**), —

I. *Convex surface = circumference of base × height of cylinder.*

II. *Total surface = convex surface + twice the area of either base.*

Exercises. — Find the convex surface, the areas of the bases, and the total surface of the following cylinders: —

1. Height $= 3^{m}$, radius of base $= 1^{m}$.

2. Height $= 96^{cm}$, diameter of base $= 60^{cm}$.

3. Height $= 4.5^{m}$, circumference of base $= 3.64^{m}$.

4. Height $= 34^{m}$, area of base $= 16.74^{qm}$.

5. Height $= 9\frac{1}{2}^{ft}$, radius of base $= 6^{ft}$.

6. Height $= h$, radius of base $= r$.

7. Height $= h$, diameter of base $= d$.

8. How many sq. yds. of sheet tin are required to cover the convex surface of a circular tower 40^{ft} high, the diameter of the base being 8^{ft}?

9. The length of a hollow tube $= 3^{m}$, the outer diameter $= 36^{cm}$, the thickness of the metal $= 2^{cm}$. Find the area of the inner convex surface.

§ **255.** THE PYRAMID. The lateral surface consists of triangles; the base is a polygon. Therefore, —

I. *Lateral surface = sum of the areas of the triangles of which it is composed.*

II. *Total surface = lateral surface + area of the base.*

Exercises. — Find the total surface of the following regular pyramids: —

1. Triangular pyramid, slant height $= 68^{cm}$, side of base $= 42^{cm}$.

2. Square pyramid, slant height $= 15^m$, side of base $= 3^m$.

3. Octagonal pyramid, slant height $= 7.06^m$, side of base $= 1.44^m$.

4. How can the above exercises be solved if the less (or greater) radius of the base is given in place of a side? (See Table, p. 221.)

5. The height of a regular triangular pyramid $= 8^m$, and the side of the base $= 2^m$. Find the total surface.

Hints.—Find less radius of base by means of the Table, p. 221, and then the slant height by means of § **142**.

6. Height of a regular hexagonal pyramid $= 18^m$, side of the base $= 1.5^m$. Find the entire surface.

7. The vertex of a pyramid with a rectangular base is vertically above the middle point of the base. Find the total surface if the height $= 6^m$, and the sides of the base are 2.4^m and 1.6^m.

8. The height of the frustum of a regular pyramid with a square base $= 16^m$, and the sides of the bases are 4^m and 2^m. Find the entire surface.

Hints.—The lateral surface consists of four equal trapezoids. Find their common altitude by means of § **142**.

§ **256.** The Cone. The developed convex surface is a circular sector (*Fig. 277*) whose arc is equal to the circumference of the cone, and whose radius is equal to the slant height. The base of the cone is a circle. Therefore (§ **216**),—

I. *Convex surface = circumference of base × half the slant height.*

II. *Total surface = convex surface + area of the base.*

Exercises.—Find the convex surface, the area of the base, and the entire surface of the following cones:—

1. Slant height $= 5^m$, radius of base $= 2^m$.

2. Slant height $= 60^{ft}$, diameter of base $= 12^{ft}$.

3. Slant height $= 12.62^m$, circumference of base $= 2.27^m$.

4. Slant height $= k$, radius of base $= r$.

5. Slant height $= k$, diameter of base $= d$.

6. Height $= 12^m$, radius of base $= 5^m$ (see § **142**).

7. Height $= h$, radius of base $= r$.

8. How many square yards of sail-cloth are required to make a conical tent 60^{ft} high, if the diameter of the base is 40^{ft}.

9. Two cones have equal bases, but one is three times as high as the other. Compare their convex surfaces.

§ **257**. Frustum of a Cone. The development of this surface is a part of a circular ring (*Fig. 283*) whose arcs AB and CD are the circumferences of the bases of the frustum, and whose distance apart is equal to the slant height of the frustum. We shall first find the area by the method of reasoning used in § **212** (a method which we shall have occasion to use again hereafter); and shall then transform the value of the area thus obtained into two other useful forms.

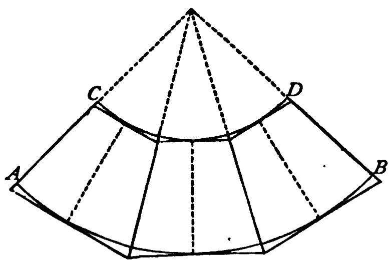

Fig. 283.

Construct a series of trapezoids, such as the three shown in *Fig. 283*, by drawing radii and parallel pairs of tangents (why are they parallel?). The common altitude of these trapezoids = distance between the arcs = slant height of frustum. By increasing the number of such trapezoids we can plainly make the sum of their areas approach the area of the frustum as nearly as we please, although it will never quite reach it. That is to say, —

Surface of the frustum = limit of the sum of the trapezoids.

By § **131** and by addition, the sum of the trapezoids = $\frac{1}{2}$ sum of their parallel sides × slant height. As the number of trapezoids is increased, the sum of their parallel sides approaches, but never quite reaches, the sum of the arcs AB and CD. Therefore, —

Limit of sum of the trapezoids = $\left\{ \begin{array}{l} \frac{1}{2}(AB + CD) \times \text{slant height} \\ \text{of frustum.} \end{array} \right.$

Therefore (Axiom I.), —

I. *Convex surface of frustum* = $\frac{1}{2}(AB \times CD) \times$ *slant height* = $\frac{1}{2}$(*sum of circumferences of bases*) × *slant height.*

The result can be put into another form, as follows: Let a frustum be generated by the revolution of the trapezoid $AXBY$ (*Fig.*

284) about the side XY. Through M, the middle point of AB, draw $MN \parallel AX$. MN is the radius of a circular section of the frustum midway between the bases. Draw $BP \parallel XY$, and intersecting MN in Q; $BY = QN = PX$ (why?) and $AP = 2\,MQ$ (why?). Put $BY = a$, $MQ = b$; then $MN = a + b$, $AX = a + 2b$. By § **209**,—

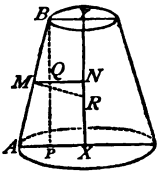

Fig. 284.

$$\text{Circumference of upper base} = 2\pi a.$$

$$\text{“ “ lower “} = 2\pi a + 4\pi b.$$

$$\text{“ “ middle section} = 2\pi a + 2\pi b.$$

Whence, $\frac{1}{2}$ (sum of circumferences of bases) $= \dfrac{4\pi a + 4\pi b}{2}$ $= 2\pi a + 2\pi b =$ circumference of middle section.

Therefore,—

II. *Convex surface of frustum = circumference of middle section × slant height.*

Draw $MR \perp AB$; $\triangle MNR \sim \triangle ABP$ (why?); therefore, $MN : MR = BP$ or its equal $XY : AB$; whence,—

$$MN \times AB = MR \times XY.$$

If we multiply both sides of this equation by 2π, we have

$$2\pi MN \times AB = 2\pi MR \times XY.$$

The first side of this equation is the value of the convex surface given in II.; the second side is the product of the circumference whose radius is MR and the height of the frustum. Therefore,—

III. *Convex surface of frustum = circumference whose radius is the length of the perpendicular erected at the middle point of the side of the frustum and terminated by the axis × height of frustum.*

Exercises.—**1.** Prove that Theorems II. and III. hold true of the convex surface of an entire cone.

Hint.—In the proof of II., make $a = 0$, because there is no upper base.

2. Prove that Theorem III. holds true of the convex surface of a cylinder.

3. How is the *total* surface of a frustum found?

Find the convex surface, and also the total surface, of the following frustums of cones:—

4. Slant height $= 11^m$, radii of bases 5^m and 3^m.

5. Slant height $= 4^{ft}$, diameter of bases 18^{in} and 10^{in}.

6. Slant height $= k$, radii of bases, a and b.

7. Slant height $= k$, radius of middle section $= r$.

8. The height of a frustum $= 12^m$, the radii of the bases are 7^m and 2^m. Find the convex surface.

Hint.—It is evident, from *Fig.* 284 and from § **142**, that slant height $= \sqrt{12^2 + (7-2)^2}$.

9. The height of a frustum $= h$, the radii of the bases are a and b. Find the convex surface.

§ **258**. The Sphere. Inscribe in the semicircle XBY (*Fig.* 285), with a radius $OX = r$, half of a regular polygon with an even number of sides (here four). Draw the less radii OM, ON, OP, OQ, and from the vertices A, B, C, let fall the perpendiculars AD, BO, CE.

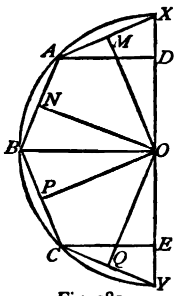

Fig. 285.

If the semi-circumference and the sides of the polygon revolve about XY, the former will describe the surface of a sphere, and the sides of the polygon will describe the convex surfaces of cones or frustums of cones. Which two sides will describe the former?

By § **257**, III. (including Exercise 1),—

$$\text{Surface described by side } XA = 2\pi OM \times DX.$$
$$\text{" \quad " \quad " } \quad AB = 2\pi ON \times OD.$$
$$\text{" \quad " \quad " } \quad BC = 2\pi OP \times EO.$$
$$\text{" \quad " \quad " } \quad CY = 2\pi OQ \times YE.$$

Adding these equations, and remembering that $OM = ON = OP = OQ$, we have,—

Total surface described by sides of polygon $= 2\pi OM \times XY$.

As the number of sides is increased, the total surface described

by the sides approaches the surface of the sphere as its limit. That is, —

$$\text{Surface of sphere} = \begin{cases}\text{limit of surface described by sides of the}\\ \text{polygon.}\end{cases}$$

As the number of sides is increased, the less radius OM approaches r as its limit. Therefore, —

$$\left.\begin{matrix}\text{Limit of surface described by}\\ \text{the sides of the polygon}\end{matrix}\right\} = 2\pi r \times XY.$$

Therefore (Axiom I.), —

Surface of the sphere $= 2\pi r \times XY = 2\pi r \times 2r = 4\pi r^2$.

Or (§§ **209**, **213**), —

I. *Surface of a sphere = circumference of a great circle × the diameter = four times the area of a great circle.*

The above method applied to a part of the semicircle, — as, for instance, the arc ABC, which, by revolving, describes a zone, — gives as the result, —

II. *Surface of a zone = circumference of a great circle × height of zone.*

Exercises. — Find the surfaces of the following spheres: —

1. Radius $= 5^m$.
2. Radius $= 3\frac{1}{2}^{in}$.
3. Radius $= r$.
4. Diameter $= 44^m$.
5. Diameter $= 12^{ft}$.
6. Diameter $= d$.

7. Find the surface of the earth, taking the diameter as 8000 miles.

8. If the surface of a sphere $= 400^{qm}$, find the radius.

9. Find the *total* surface of a hemisphere if the radius $= 6^m$.

10. Find the total surface of a hemisphere if the radius $= r$.

11. How much leather will it take to cover a base-ball if the diameter $= 4\frac{1}{2}^{in}$?

12. If the radius of a sphere $= 8^m$, find the area of a zone the height of which $= 2^m$.

13. Height of a cylinder = diameter of its base. Compare its total surface with that of the largest sphere that can be inscribed in it.

Ans. Surface of the sphere $= \frac{2}{3}$ surface of the cylinder.

NOTE. — This curious fact was discovered by Archimedes more than 200 B.C.

II.—Volumes of Bodies.

§ **259**. In order to compare the sizes or magnitudes of bodies, we select a body of known size as the common term of comparison or unit, and then find how many times this unit is contained in other bodies.

Definition. — *The number of times the unit is contained in a body, followed by the name of the unit, is called the* VOLUME *of the body.*

If, for instance, the cubic inch is the unit, and one body will contain it four times while another body will contain it twelve times, the volumes of the bodies are four cubic inches and twelve cubic inches; and one of the bodies is exactly three times as large as the other.

The most convenient units of volume are *cubes whose edges are equal to the units of length.*

The CUBIC INCH (cub. in.), the CUBIC FOOT (cub. ft.), and the CUBIC YARD (cub. yd.), are such units. The BUSHEL and the GALLON are not such units. The bushel contains 2150.42 cub. in., the (United States) *liquid* gallon contains 231 cub. in., and the (United States) *dry* gallon contains 268.8 cub. in.

In the metric system, the units of volume are always cubes whose edges are units of length. The three most used are the CUBIC METER (cbm.), the CUBIC DECIMETER (cdm.), and the CUBIC CENTIMETER (ccm.).

The cubic decimeter is often called a LITER (l.). A HECTOLITER (hl.) = 100 liters.

The cubic meter, when used for measuring wood, is called a STERE.

NOTE. — 1 cubic foot	= 0.0283 cubic meter.	1 cubic meter	= 35.3156 cubic feet.
1 bushel	= 35.24 liters.	1 liter	= 0.02838 bushel.
1 liquid gallon	= 3.785 liters.	1 liter	= 0.264 gallon.

Exercises. — 1. The bushel contains 4 *pecks.* How many cubic inches in one peck?

2. The gallon of either kind is divided into four *quarts*, the quart into two *pints*, the pint into four *gills*. How many cubic inches in the quart, pint, and gill, both liquid and dry ?

3. Reduce 6000 cubic feet to cubic meters. 6000 cubic meters to cubic feet.

4. If I pay $1.20 per bushel for wheat, what must I ask per hectoliter, in order to make a profit of 25 per cent ?

5. A hectoliter of wine will fill how many pint bottles ?

§ **260.** To *measure* the volume of a body is to find how many times it will contain the unit of volume.

The volume of a liquid body may be found directly by repeatedly filling a vessel which is known to hold exactly a unit; as, for example, a quart pot or a gallon jug. But this method could not be applied to bodies of fixed shape, such as a stick of timber or a cannon-ball.

Geometry teaches how volumes are measured indirectly, by showing that they depend upon the lengths of certain lines (for instance, the length, breadth, and height of a parallelopiped), and can, therefore, be found by performing simple operations upon the numbers which express the lengths of these lines (compare § **126**).

We shall deduce a general rule for computing the volume of any prism or cylinder by treating as special cases,—

(*i.*) The rectangular parallelopiped;

(*ii.*) The right parallelopiped;

(*iii.*) The oblique parallelopiped;

(*iv.*) The triangular prism;

(*v.*) Any prism;

(*vi.*) The cylinder;

and another general rule for computing the volume of any pyramid or cone by treating as special cases,—

(*i.*) The triangular pyramid;

(*ii.*) Any pyramid;

(*iii.*) The cone.

§ **261.** THE RECTANGULAR PARALLELOPIPED. In the rectangular parallelopiped AP (*Fig. 286*) let the edges be $AB = 5^m$, $AD = 3^m$, $AM = 7^m$. Then the base $ABCD$ contains 15^{qm} (§ **128**); and upon each square meter of the base we can construct a pile of 7 cubic meters, such as DQ, so that the solid must contain exactly $7 \times 15 = 105$ cubic meters.

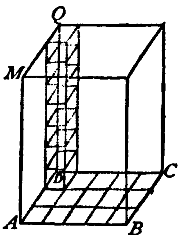

Fig. 286.

A like result would be obtained if the edges had any other values. If there were fractions of a meter, we would proceed as shown in finding the area of a square (§ **127**). Therefore, the number of units of volume in a rectangular parallelopiped is found by *multiplying the number of units of area in the base by the number of linear units in the height.* Or, more briefly, —

Volume of a rectangular parallelopiped = base × height.

If a, b, and c denote the length, the breadth, and the height of the solid, then the base $= a \times b$, and therefore the volume $= a \times b \times c$.

In the case of the cube, $a = b = c$; therefore, —

Volume of a cube $= a \times a \times a = a^3$.

Here a^3 is a short way of expressing that a is taken as a factor three times, *or is raised to the third power.* Hence the third power of a number is often called the *cube* of the number.

Remark. — Since the edge of a cubic meter = 10 decimeters, therefore, 1 cubic meter $= 10 \times 10 \times 10 = 1000$ cubic decimeters (liters). For a like reason, 1 cubic decimeter = 1000 cubic centimeters. In general, —

The ratio between two units of volume is the cube of the ratio between the corresponding units of length.

Exercises. — Find the volumes of the following cubes: —

1. Edge $= 8^m$. **3.** Edge $= 72^{cm}$. **5.** Area of base $= 36^{qm}$.
2. Edge $= 3.17^m$. **4.** Edge $= 9\frac{3}{8}^{in}$. **6.** Area of base $= S$.

Find the volumes of the following rectangular parallelopipeds:—

7. Length $= 4.5^{m}$, breadth $= 7^{m}$, height $= 3.4^{m}$.

8. Length $= l$, breadth $= b$, height $= h$.

9. Length $=$ breadth $= l$, height $= h$.

10. Area of base $= 81^{qm}$, height $= 4^{m}$.

11. Area of base $= S$, height $= h$.

12. How many cubes, each with an edge $= 1^{dm}$, can be put into a box 12^{dm} by 8^{dm} by 6^{dm}, inside measurement?

13. Find the number of cubic decimeters of wood in a board 5^{m} long, 0.6^{m} wide, 0.03^{m} thick. What kind of a solid is the board?

14. How high must a box 5^{dm} long and 2^{dm} wide be to hold 30 liters?

15. Volume of a cube $= 74{,}088^{ccm}$. Find the edge.

16. Volume of a cubical tank $= 474.552^{ccm}$. Find its depth.

17. Volume of a cube $= V$. Find the edge.

18. Two dimensions of a rectangular parallelopiped are 9.37^{m} and 5^{m}. Find the third dimension.

19. A room is 8^{m} long, 6^{m} wide, 2.5^{m} high. How many cubic meters does it contain? How much more space would it inclose if the floor had the same perimeter, but were in the shape of a square?

20. How many stones 3^{dm} long, 1.5^{dm} wide, 0.75^{dm} thick, are required to build a wall 40^{m} long, 3^{m} high, and 0.5^{m} thick?

21. How many liters of water have fallen in a garden 42^{m} long and 32^{m} wide, if a vessel standing in it is filled to the height of 12^{cm}?

22. How many hogsheads will a tank 10^{ft} by 12^{ft} by 9^{ft} hold? (1 hogshead $= 63$ gallons.)

23. Why does one cubic foot contain 1728 cubic inches.

§ **262.** The Right Parallelopiped. — If from the right parallelopiped AG (*Fig. 287*) we take away the prism BG, by passing through the edge BF a plane $BFMN$ perpendicular to the base, and then place this prism on the left in the position AH, the right parallelopiped will be transformed into an equivalent rectangular parallelopiped having the same height and an equivalent base (§ **129**). Therefore, by § **261**,—

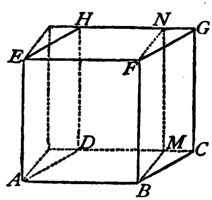

Fig. 287.

Volume of a right parallelopiped $=$ *base* $\times$ *height.*

Since area $ABCD = AB \times BM$, therefore, *the volume of the right parallelopiped* $AG = AB \times BM \times BF = AB \times BF \times BM$ = area $ABEF \times BM$ = *a rectangular face taken as base* $\times$ *the corresponding height.*

Exercises. — 1. Find the volume of a right parallelopiped if the height = 11^m, and the base is a parallelogram having the length 12^m and the altitude 5^m.

2. A wall 20^m long, 3^m high, is 0.8^m thick at the bottom and 0.4^m thick at the top. Find how many cubic meters it contains.

Hint. — Consider the wall a right parallelopiped having for base a trapezoid whose parallel sides are 0.8^m and 0.4^m.

§ **263.** THE OBLIQUE PARALLELOPIPED. — Let AG (*Fig. 288*) be an oblique parallelopiped. Through the corner E pass a plane perpendicular to the edge EF, cutting from the parallelopiped the solid $AMDNEHO$, and place this solid on the right in the position $BPCQFGR$. By this operation, the oblique parallelopiped is transformed into an equivalent right parallelopiped, having the same height, and a rectangular base $EFRO$ equivalent to the base $EFGH$ of the oblique parallelopiped (§ **129**). Now the volume of the right parallelopiped = its base $EFRO \times$ the height (§ **262**); therefore, —

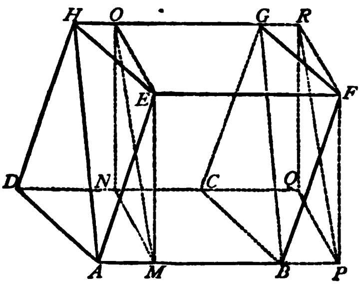

Fig. 288.

Volume of the oblique parallelopiped = base $EFGH \times$ *height.*

§ **264.** THE TRIANGULAR PRISM. — If through the edges AB and GH of the oblique parallelopiped AG (*Fig. 288*) we pass a plane, it will divide the solid into two oblique triangular prisms. The same plane divides the equivalent right parallelopiped into two right triangular prisms. These right prisms are equivalent, each to its

corresponding oblique prism, because each is formed by cutting off (by the plane through $E \perp$ the edge EF) a portion of the oblique prism, and transferring it to the right of the figure. Now these right prisms are also equivalent to each other; for, if one of them is inverted, it is then clear that it has precisely the same shape as the other, and that its faces are equal and parallel to those of the other. Therefore, the two oblique prisms are equivalent, and hence each is half of the parallelopiped. Take $ADEH$ as the base of the parallelopiped; then the distance between the planes $ADEH$ and $BCFG$ is its height, the two prisms have the same height, and their bases are the triangles AEH and ADH, each of which is half the parallelogram $ADEH$. So that the volume of either prism (half the volume of the parallelopiped) is equal to its base (half the base of the parallelopiped) × its height. In general, —

Volume of a triangular prism = its base × its height.

Exercises. — Find the volume of a regular triangular prism, having given, —

1. Height × 75^{cm}, area of base = 140^{ccm}.

2. Height = h, area of base = S.

3. Height = 8.5^{m}, one side of the base = 3^{m}.

4. Height = h, one side of the base = a.

5. The bases of a triangular prism 80^{cm} high are isosceles right triangles, a leg of which = 12^{cm}. Find the volume of the prism.

6. If the volume of a triangular prism with isosceles right triangles for bases = 600^{cbm}, and the height = 14^{m}, find (*i.*) area of the base, (*ii.*) perimeter of the base.

§ **265.** Any Prism. — Any prism (*Fig. 289*) can be divided into triangular prisms by passing planes, as shown in the figure, through one of the lateral edges. The triangular prisms have the same height as the entire prisms, and the sum of their faces is equal to the base of the entire prism. Therefore, by § **264** and by addition, it follows that the —

Volume of any prism = its base × its height.

Hence, *any two prisms having equivalent bases and equal heights are equivalent.*

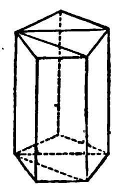

Fig. 289.

Exercises. — Find the volumes of the following regular prisms: —

1. Hexagonal prism, height $= 17^{m}$, side of base $= 2^{m}$.

2. Hexagonal prism, height $= h$, side of base $= a$.

3. Octagonal prism, height $= 85^{cm}$, side of base $= 10^{cm}$

4. Octagonal prism, height $= h$, side of base $= a$.

§ **266.** THE CYLINDER. — If we inscribe in the bases of a cylinder regular polygons of the same number of sides, so that their sides, taken pair by pair, are parallel, and then pass planes through each pair of sides, these planes with the polygons will form a prism *inscribed* in the cylinder (*Fig. 290*). Now the greater the number of lateral faces in the inscribed prism, the nearer does its volume approach that of the cylinder; that is to say, —

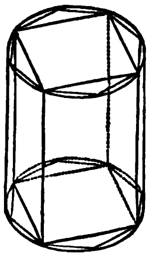

Fig. 290.

Volume of cylinder = limit of volume of inscribed prism.

The volume of the prism = its base × its height; and its base has the base of the cylinder for its limit. Therefore, limit of the volume of the inscribed prism = base of cylinder × its height.

Hence (Axiom I.), —

Volume of the cylinder = its base × its height.

Exercises. — Find the volumes of the following cylinders: —

1. Height $= 6^{m}$, area of base $= 42^{qm}$.

2. Height $= 8^{m}$, radius of base $= 2^{m}$.

3. Height $= 64^{cm}$, diameter of base $= 16^{cm}$.

4. Height $= 7\frac{1}{2}^{ft}$, circumference of base $= 44^{in}$.

5. Height $= h$, radius of base $= r$.

6. Height $= h$, radius of base $= d$.

7. The volume of a cylinder $= 200^{cbm}$, the height $= 13^{m}$. Find the radius of the base.

8. The volume of a cylinder $= V$, the height $= h$. Find the radius of the base.

9. A cylindrical pail holding exactly one liter is 18^{cm} high. Find the diameter of the base.

10. A hollow iron tube is 6^m long, the outer diameter $= 92^{cm}$, the inner diameter $= 68^{cm}$. How many cubic decimeters of iron are there ?

11. How long is an iron tube, if the inner diameter $= 3.5^{cm}$, the outer circumference $= 15^{cm}$, the volume $= 120^{ccm}$?

12. A vessel in the shape of a cylinder is to be made from sheet-iron 2^{cm} thick. The vessel is to be 1^m high, and to hold 100 liters. What must be its outer diameter ?

§ **267.** THE TRIANGULAR PYRAMID. — Let $S-ABC$ and $S-MNO$ (*Fig. 291*) be two triangular pyramids standing on the same horizontal plane, and having equivalent bases ABC and MNO, and equal heights. Divide the common height into any number of equal parts, and through the points of division pass planes parallel to the bases of the pyramids. Since the bases are equivalent, it follows, from the mode of generation of a pyramid (§ **246**), that any two sections, as DEF and PQR, situated at the same distance from the bases, will also be equivalent.

In the two pyramids construct triangular prisms, such as $DEFGHI$ and $PQRTUV$, having for upper base one of the sections; for lateral edges, lines parallel to SA or SM; and for a common height, the distance apart of two sections. It follows, from § **265**, that these prisms will be equivalent, taken pair by pair, in the two pyramids; therefore, the sums of their volumes will be equal, no matter how many or how few of them there may be. But, the greater their number the nearer the sums of their volumes approach the volumes of the two pyramids; that is to say, —

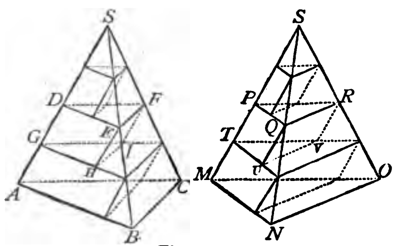

Fig. 291.

Volume of each pyramid = limit of sum of volumes of the inscribed prisms.

Now the sums of these volumes remain always equal; therefore their limits, or the volumes of the pyramids, must also be equal.

Hence, *two triangular pyramids having equivalent bases and equal heights are equivalent.*

Now let $S-ABC$ (*Fig. 292*) be any triangular pyramid. If we pass a plane through $BC \parallel SA$, and a plane through $S \parallel ABC$, we form a triangular prism $ABCSPQ$. This prism is divided by the plane SBC, and a plane passed through C and SP, into three triangular pyramids, $S-ABC$, $P-SBC$, and $C-SPQ$. Of these, $S-ABC$ and $C-SPQ$ are equivalent because they have equivalent bases and equal heights. Also, $P-SBC$ and $C-SPQ$ are equivalent, because we may take S for the common vertex, and then they have equal heights (the distance from S to the plane $BCPQ$) and equivalent bases (the equal triangles PBC and PCQ). Therefore, each pyramid $= \frac{1}{3}$ the prism. Now the volume of the prism = base × height; therefore, the volume of the pyramid $S-ABC$ (which has the same base and the same height as the prism) $= \frac{1}{3}$ base × height. Therefore, —

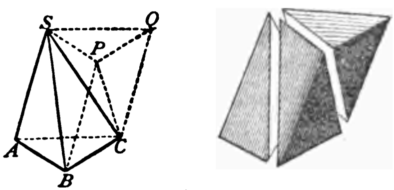

Fig. 292.

Volume of a triangular pyramid $= \frac{1}{3}$ *its base × its height.*

Exercises. — **1.** Height of a regular triangular pyramid $= 15^m$, one side of the base $= 2^m$. Find the volume.

2. Height of a regular triangular pyramid $= h$, one side of the base $= a$. Find the volume.

3. A triangular pyramid 90^{cm} high has an isosceles right triangle for base, one leg of which $= 40^{cm}$. Find the volume of the prism.

4. A triangular pyramid with an isosceles right triangle for base is 3.4^m high. Its volume $= 80^{cm}$. Find (*i.*) the area of the base; (*ii.*) the perimeter of the base.

5. The frustum of a regular triangular pyramid is 4^m high, and two homologous sides of the bases are 2^m and 1.25^m. Find its volume. (See § **249.**)

§ **268.** Any Pyramid.—Any pyramid whatever can be divided into triangular pyramids by passing planes through one of the lateral edges, as shown in *Fig. 293*. These triangular pyramids all have the same height as the entire pyramid, and the sum of their bases = the base of the entire pyramid.

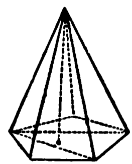

Fig. 293.

Therefore, —

Volume of any pyramid $= \frac{1}{3}$ *base* $\times$ *height.*

Exercises.—1. Find the volume of a regular hexagonal pyramid, if the height $= 36^{cm}$ and a side of the base $= 8^{cm}$.

2. Find the volume of a regular octagonal pyramid, if the height $= 44^{m}$ and a side of the base $= 4^{m}$.

3. Solve Exercise 1, if the height $= h$ and a side of the base $= a$.

4. Solve Exercise 2, if the height $= h$ and a side of the base $= a$.

5. The frustum of a regular hexagonal pyramid is 2.7^{m} high, and two homologous sides of the bases are 1.2^{m} and 70^{cm}. Find the volume.

6. A vessel has the shape of the frustum of a regular four-sided pyramid. It is 18^{cm} high, and the outer edges of its bases are 24^{cm} and 16^{cm}. How many liters of water will it hold if the material of which it is made is 2^{cm} thick?

§ **269.** The Cone.—Inscribe in the base of a cone (*Fig. 294*) a regular polygon, and pass planes through the sides of this polygon and the vertex of the cone. These planes with the polygon form an inscribed regular pyramid whose volume approaches the nearer to that of the cone the greater the number of sides in the regular polygon; that is, —

Volume of cone = limit of volume of inscribed pyramid.

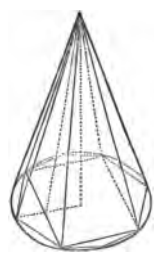

Fig. 294.

But volume of any pyramid $= \frac{1}{3}$ its base $\times$ its height. And in this case the base has the base of the cone for its limit. Therefore, limit of volume of inscribed pyramid $= \frac{1}{3}$ base of cone $\times$ height. Therefore (Axiom I.), —

Volume of a cone $= \frac{1}{3}$ *its base* $\times$ *its height.*

Exercises. — Find the volumes of the following cones: —

1. Height $= 15^m$, area of base $= 60^{qm}$.

2. Height $= 12^m$, radius of base $= 3^m$.

3. Height $= h$, radius of base $= r$.

4. Height $= 74^{cm}$, diameter of base $= 4.6^m$.

5. Height $= h$, diameter of base $= d$.

6. Height $= 6^{ft}$, circumference of base $= 1\frac{3}{8}^{ft}$.

7. Slant height $= 5^m$, radius of base $= 3^m$.

8. Slant height $= k$, radius of base $= r$.

9. Find the volume of a cone whose slant height = diameter of the base $= 2^m$.

10. Volume of a cone $= 800^{cbm}$, radius of base $= 1.8^m$. Find the height.

11. Find the volume of the frustum of a cone, if the radii of the bases are 7.5^{cm} and 5^{cm}, and the height 25^{cm}. (See § **249**.)

12. A vessel has the shape of the frustum of a cone 1^m high, and the inner circumferences of the bases are 5^m and 4^m. How many cubic meters of water will it hold?

13. What part of a cone remains if a cone two-fifths as high as the entire cone is cut off?

14. How many gallons of water will a vessel hold which has the shape of the frustum of a cone, the height being 3^{ft}, the diameters of the bases 18^{in} and 12^{in}?

15. A Dutch windmill in the shape of the frustum of a cone is 12^m high. The outer diameter at the bottom and the top are 16^m and 12^m; the inner diameters 12^m and 10^m. How many cubic meters of stone were required to build it?

16. The chimney of a factory is 32^m high, and the outer circumferences of its bottom and top are 20^m and 14^m, while the bore is, throughout, 2^m wide. How many cubic meters of brick does it contain?

§ **270.** The results obtained in §§ **261–266** may be summed up in one general rule, as follows: —

Volume of a prism or a cylinder = base × height.

And the results obtained in §§ **267–269** may be summed up in one general rule, as follows: —

Volume of a pyramid or a cone = $\frac{1}{3}$ base × height.

§ **271.** THE SPHERE.—In order to find the volume of a sphere, conceive a series of pyramids formed, having for a common vertex the centre of the sphere, and for bases the faces of a polyhedron circumscribed about the sphere (that is, a polyhedron whose faces touch the surface of the sphere, each in a single point). By increasing the number of these pyramids, we can make their sum approach the volume of the sphere as nearly as we please; or, in other words,—

Volume of sphere = limit of sum of the pyramids.

Now each pyramid = $\frac{1}{3}$ height $\times$ base, and their common height = radius of the sphere; therefore, sum of the pyramids = $\frac{1}{3}$ radius of sphere $\times$ sum of the faces. The limit of the sum of the faces (when their number is increased more and more) = the surface of the sphere. Therefore, limit of the sum of the pyramids = $\frac{1}{3}$ radius $\times$ surface of the sphere. Therefore (Axiom I.),—

Volume of a sphere = $\frac{1}{3}$ *radius* $\times$ *surface of sphere.*

Exercises.—**1.** Find the volume of a sphere if the radius = 6^{m}.

2. Find the volume of a sphere if the radius = r.

3. Find the volume of a sphere if the diameter = 14^{cm}.

4. Find the volume of a sphere if the diameter = d.

5. Find the volume of a sphere if the circumference of a great circle = 10^{ft}.

6. Required the volume of the largest sphere that can be turned from a cubical block of wood whose edge = 1^{dm}. How much of the wood is lost?

7. The volume of a sphere = 180^{cbm}. Find the radius.

8. The volume of a sphere = V. Find the radius.

9. Find the volume of the largest sphere that can be turned from a cylinder of wood whose diameter = its height = 12^{cm}.

10. The diameter of a cylinder = its height = h. Compare the volume of this cylinder with that of the largest sphere that can be inclosed in it.

Ans. Volume of the sphere = $\frac{2}{3}$ volume of the cylinder.

NOTE.—This curious fact was discovered by Archimedes about 230 B.C.

11. Find the difference between the volumes of two concentric spheres whose radii are 4^{m} and 5^{m}.

12. In two concentric spheres, the volume of the larger = twice that of the smaller. If the radius of the smaller = 1^{m}, find the radius of the larger.

§ **272.** The volume of any body, whatever be its shape, may be found by *weighing* it, and then *dividing the weight by the weight of one unit of volume.* The quotient will be the number of units of volume in the body.

In the metric system, the most common units of weight are, —

The GRAM (g.) = *the weight of* 1^{ccm} *of pure water* (*at* $4°$ *C.*).
The KILOGRAM (kg.) = *weight of* 1^{cdm} *of pure water* (*at* $4°$ *C.*).

Since $1^{cdm} = 1000^{ccm}$, 1 kilogram = 1000 grams.

The ratio of the weight of any substance to the weight of the *same* volume of pure water is called the SPECIFIC GRAVITY of that substance.

SPECIFIC GRAVITIES OF SOME COMMON SUBSTANCES.

Platinum	22.00	Brass (sheet)	8.40	Boxwood	1.20
Gold	19.35	Steel	7.80	Ice	0.92
Mercury	13.60	Iron (cast)	7.25	Oak	0.90
Lead	11.35	Zinc	7.19	Alcohol	0.80
Silver	10.51	Marble	2.70	Pine	0.57
Copper	8.80	Glass	2.60	Cork	0.24

The gram and the kilogram being by definition the weights of unit volumes of water, it follows that the above numbers express the weights of 1^{ccm} of the substance in grams, or of 1^{cdm} (1 liter) of the substance in kilograms.

Whence it is clear that in the metric system, —

$$\textit{Volume of a body} = \frac{\textit{its weight}}{\textit{its specific gravity}};$$

the volume being expressed (*i.*) in cubic centimeters, or (*ii.*) in cubic decimeters (liters), according as the weight is expressed (*i.*) in grams or (*ii.*) in kilograms.

The most important English units of weight are the POUND (lb.) avoirdupois, the OUNCE (oz.) = $\frac{1}{16}^{lb}$, and the TON = 2000^{lb}.

They are connected (nearly enough for practical purposes) with the weight of unit volume of water by the relation,—

Weight of 1 *cubic foot of water* $= 62\frac{1}{2}^{\text{lbs}} = 1000^{\text{oz}}$.

The weight in *pounds* of a cubic foot of any other substance is found by multiplying its specific gravity by $62\frac{1}{2}$, and the weight in *ounces* by multiplying its specific gravity by 1000.

NOTE.—$28\frac{3}{4}^{\text{g}} = 1^{\text{oz}}$ (nearly); $1^{\text{kg}} = 2.2^{\text{lbs}}$ (nearly).

Exercises.—**1.** Find the volume of a piece of granite which weighs 4^{kg}.

2. A piece of lead weighs 24^{kg}. Find its volume.

3. What is the volume of 84^{g} of silver?

4. Required the edge of a cubical vessel which will hold 2000^{lbs} (1 ton) of mercury.

5. Find the edge of a marble cube weighing 184^{kg}.

6. An iron cylinder 2.5^{m} long weighs 680^{kg}. Find its diameter.

7. Find the edge of a cube of cork which weighs the same as an iron ball 22^{cm} in diameter.

8. How many liters will a vessel hold if the vessel weighs when empty 1.5^{kg}, and when full of water 1.5^{kg}?

9. An oak cone 40^{cm} high weighs 2^{kg}. Find its diameter.

10. What is the diameter of a cannon ball which weighs 64^{lbs}?

11. How can the weight of a body be found if its volume and its specific gravity are known?

12. Find the weight of 1^{hl} of water, 1^{hl} of alcohol, and $1^{\text{cub ft}}$ of mercury.

13. Find the weight of a rectangular parallelopiped of copper, 0.34^{m} long, 0.11^{m} wide, 0.03^{m} thick.

14. Find the weight of an oak cone, if the radius of the base $= 32^{\text{cm}}$ and the height $= 90^{\text{cm}}$.

15. Find the weight of a hollow brass cube, if the inner diameter $= 3^{\text{dm}}$ and the thickness of the brass $= 1^{\text{cm}}$.

16. If a cube, whose edge $= 8^{\text{cm}}$, weighs 50^{g}, find the weight of the largest cylinder that can be turned from it. Also the weight of the largest cone. Also the weight of the largest sphere.

17. How can the specific gravity of a body be found, if its weight and its volume are known?

18. A cubical block of pine wood, an edge of which $= 12^{\text{cm}}$, weighs 1^{kg}. Find the specific gravity of the wood.

19. If a mass of ice containing 270^{cbm} is known to weigh $229{,}000^{\text{kg}}$, find the specific gravity of ice.

20. Find the specific gravity of cast iron, if $2\frac{1}{8}^{\text{cu ft}}$ weigh 953^{lbs}.

§ **273**. *Equivalent* solids have the same volume (size); *similar* solids, the same shape; *equal* solids, the same volume and the same shape.

Equal polyhedrons must fulfil three conditions: —

(*i.*) They must have the same number of faces, *equal* each to each.

(*ii.*) Their homologous edges must be *equal.*

(*iii.*) Their homologous solid angles must be *equal.*

Similar polyhedrons must also fulfil three conditions: —

(*i.*) They must have the same number of faces *similar* each to each.

(*ii.*) Their homologous edges must be *proportional.*

(*iii.*) Their homologous solid angles must be *equal.*

In both cases, there are the same number of edges and also of corners.

Conversely, if *all three* of the above conditions hold true of two polyhedrons, then these polyhedrons are equal or similar, as the case may be.

There are four important classes of similar solids: —

(*i.*) *All cubes and regular polyhedrons of the same name are similar.*

(*ii.*) In order that two regular prisms or two regular pyramids may be similar, they must have the same number of faces, and their homologous lateral edges must be proportional to the homologous sides of their bases. But the homologous lateral edges in the prisms are obviously proportional to the heights, and in the pyramids they are easily proved to be proportional to the heights, by reasoning as in § **249**; and the homologous sides of the bases are (by § **167**) proportional to the less radii of the bases. Therefore, *two regular prisms or pyramids with the same number of sides are similar, if their heights are proportional to the less radii of their bases.*

(*iii.*) By regarding the cylinder or the cone as the *limit* of the inscribed regular prism or regular pyramid, when the number of its sides is increased more and more (§§ **266, 269**), it becomes evident that *two cylinders or two cones are similar, if their heights are proportional to the radii of their bases.*

(*iv.*) *All spheres are similar.*

§ **274.** If the edge of a cube $= a$, the (total) surface $= 6a^2$, and the volume $= a^3$.

Now, if we have other cubes whose edges are $2a$, $3a$, $4a$, etc., their surfaces will be 4, 9, 16, etc., times that of the first cube, since $(2a)^2 = 4a^2$, $(3a)^2 = 9a^2$, $(4a)^2 = 16a^2$, etc.; and their volumes will be 8, 27, 64, etc., times that of the first cube, since $(2a)^3 = 8a^3$, $(3a)^3 = 27a^3$, $(4a)^3 = 64a^3$, etc.

Bearing in mind that 4, 9, 16, etc., are the *squares*, and 8, 27, 64, etc., the *cubes*, of 2, 3, 4, etc., we see that in cubes the surface is proportional to the *square*, and the volume proportional to the *cube*, of the edge.

What has here been shown to be true of the surface and volume of the cube in respect to its edges, can be shown to be true of the surface and volume of all similar solids in respect to any homologous lines (edges, heights, radii, diameters, etc.).

That is to say, —

I. *The surfaces of two similar solids are proportional to the* SQUARES *of any two homologous lines.*

II. *The volumes of two similar solids are proportional to the* CUBES *of any two homologous lines.*

Exercises. — 1. Compare the surface and the volume of a cube whose edge $= 2^m$ with those of a cube whose edge $= 1^m$.

2. The edge of a cube $= 1^m$. Find the edge of a cube, (*i.*) having twice the surface, (*ii*) having twice the volume of the first cube.

3. Required the depth of a cubical tank which will hold 64 times as much as a similar tank whose depth $= 3^m$.

4. Compare (*i.*) the surfaces, (*ii.*) the volumes, of the rectangular parallelopipeds whose edges are a, b, c, and $3a$, $3b$, $3c$.

5. Given two similar cylinders whose diameters are d and $4d$, and whose heights are h and $4h$. Prove that Laws I. and II., stated above in italics, hold true of them.

6. Same exercise applied to two similar cones.

7. Given two spheres with radii r and $5r$. Prove that the above laws hold true of their surfaces and their volumes.

8. The edge of a polyhedron $= 8^{m}$, and the homologous edge of a similar polyhedron $= 5^{m}$. How many times will the first polyhedron contain the second?

9. A box is 90^{cm} long, 58^{cm} wide, and 43^{cm} deep. Find the dimensions of a similar box twice as large.

10. Find the volume of the box in the last exercise if its three dimensions are doubled.

11. Find its volume if one dimension is doubled. If two dimensions are doubled.

12. I wish to make an iron cylinder, similar to a cylinder whose diameter $= 26^{cm}$ and whose height $= 64^{cm}$, but three times as large. What must be its dimensions? How many kilograms of iron are required?

13. How far from the vertex of a cone 50^{m} high must a plane parallel to the base be passed in order that the cone cut off may be exactly $\frac{1}{1000}$ of the entire cone?

14. Divide into two equivalent parts, by a plane parallel to the base, a cone 65^{cm} high, and having a base whose diameter $= 30^{cm}$.

15. If the radius of a sphere $= 37^{cm}$, what will be the radius of a sphere ten times as large?

16. I have a cylindrical boiler 60^{ft} long and 8^{ft} in diameter, and I wish another boiler similar in shape but $\frac{3}{4}$ as large. What must be its dimensions?

17. If the edge of a regular tetahedron $= 1^{m}$, it can be proved that the surface $= 1.732^{qm}$, and the volume $= 0.1178^{cbm}$; find the surface and the volume if the edge $= 7^{m}$.

18. If the edge of a regular octahedron $= 1^{m}$, it can be proved that the surface $= 3.4641^{qm}$, and the volume $= 0.4714^{cbm}$; find the surface and the volume if the edge $= 4^{cm}$.

III.—Exercises and Applications.

§ 275. Prisms and Cylinders.

1. How many steres of wood in a cubical pile one edge of which $= 12^{m}$?

2. Total surface of a cube $= 80^{qcm}$; find its volume.

3. Total surface of a cube $= S$; find its volume.

4. Volume of a cube $= 200^{cbm}$; find its surface.

5. Volume of a cube $= V$; find its surface.

6. A diagonal on one face of a cube $= 12^{cm}$; find the volume of the cube.

7. Edge of a cube $= 4^{cm}$; find the diagonal *through* the cube.

8. Edge of a cube $= a$; find the diagonal through the cube.

9. A diagonal through a cube $= 30^{cm}$; find the edge.

10. Find the edge of a cork cube which will weigh the same as 1^{ccm} of iron. (See Table, page 298.)

11. Volumes of two cubes are as $8 : 27$; what is the ratio of their edges?

12. Find the edge of a cube half as large as a cube whose edge $= 16^{cm}$.

13. Edges of two cubes are 6^{cm} and 12^{cm}; find the edge of a cube equal to their sum.

14. Edges of two cubes are 12^{cm} and 16^{cm}; find the edge of a cube equal to their difference.

15. Find the edge of a cubical tank which holds 1000^{hl} of water.

16. A water tank has a rectangular base 3.2^{m} by 2.5^{m}. How many liters of water does it contain when the height of the water $= 1.24^{m}$?

17. A tank has a square base one edge of which $= 3^{m}$. How deep is the water when the tank contains 1800^{l}?

18. A cubical reservoir whose edge $= 24^{m}$ contains how many hectoliters of water when the water is 4^{m} deep?

19. If water enters the above reservoir, through a tube, at the rate of 160^{l} per minute, in what time will it be full?

20. If it is found that in 24 hours the reservoir loses by evaporation a layer 2^{cm} in depth, how many liters does it lose?

21. What time will be required to fill the reservoir, taking account of evaporation?

22. How large a field can be irrigated by the reservoir full of water if 800^{cbm} of water are required for every hectare?

23. If the dimensions of a rectangular parallelopiped are a, b, and c, find the length of the diagonal from an upper corner to the lower opposite corner?

24. What is the cost of digging a cellar 11.5^{m} long, 8.8^{m} wide, and 2.62^{m} deep, at the rate of $0.80 per cubic meter?

25. What is the weight of a marble block 2.75^{m} long, 0.35^{m} wide, and 2.25^{m} thick?

26. *A body weighs less in water than in air by exactly the weight of its own volume of water.* If a body 1.2^{m} long, 0.62^{m} wide, and 0.4^{m} thick weighs 400^{kg} in air, what will it weigh in water?

27. Find the specific gravity of the body in the last exercise.

28. How much will a block of marble 2^{m} long, 1.6^{m} wide, and 0.8^{m} thick weigh under water?

29. Which will weigh the most in water, an iron cube whose edge $= 12^{cm}$ or a rectangular parallelopiped of lead 15^{cm} long, 10^{cm} wide, and 8^{cm} thick?

30. In a room 6.2^{m} long, 4.75^{m} wide, and 2.88^{m} high are 6 persons. In what time will the air in the room become unfit for respiration if each person breathes 20 times a minute, and at each breath vitiates 400^{ccm} of air?

31. What is the weight of the air in the above-mentioned room? (Specific gravity of the air $= 0.0013$.)

32. What must be the height of a box 2.5^{m} long and 2^{m} wide, that it may hold 22.5^{cbm}?

33. A rectangular vessel 48^{cm} long and 36^{cm} wide contains 56 liters of water. What is the depth of the water?

34. Into a corn-bin 8^{m} long and 5.2^{m} wide, 56^{hl} of corn are put; find the height.

35. The total surface of a cube $= 54^{km}$; find the sum of its edges.

36. Ten leaden cubes, each with an edge $= 20^{cm}$, are melted into one cube; find its edge.

37. If the dimensions of a trunk are 1^{m}, 15^{m}, and 2^{m}, find the dimensions of a similar trunk that will hold eight times as much.

38. Find the dimensions of a similar trunk holding twice as much.

39. A wooden trough is to be made, whose inner dimensions are to be, length 3.35^{m}, breadth 1.6^{m}, depth 1^{m}; the ends are to be 10^{cm} thick, the sides 8^{cm}, and the bottom 12^{cm}. (*i.*) How many cubic metres of wood are required? (*ii.*) How many liters will the trough hold? If the trough is made of oak wood, what will it weigh? (*iii.*) when empty? (*iv.*) when full of water?

40. What will it cost to dig a ditch around a field in the shape of a square, containing 4 hectares, if the ditch is to be 2^{m} deep, 1^{m} wide at the bottom, 1.4^{m} at the top, and the price is $0.25 per cubic metre of earth thrown out?

41. How many cubic metres of manure are required to cover with a layer 2^{cm} thick a triangular field whose base $= 840^{m}$ and altitude $= 320^{m}$?

42. If 7800^{hl} of gravel are spread upon a rectangular meadow 560^{m} by 280^{m}, how much gravel is required to make a layer of the same thickness upon a trapezoidal field whose parallel sides are 920^{m} and 740^{m}, and altitude 250^{m}?

43. Through an orifice in the side of a vessel 6^{cm} square, water flows at the rate of 16^{m} per second. How many hectoliters will flow out in one minute?

44. Through a rectangular orifice 12^{cm} by 8^{cm} water flows at the rate of 20^{m} per second. How much water will flow out in one hour?

45. If one bushel of wheat makes 48 lbs. of flour, and one barrel of flour contains 196 lbs., how many barrels of flour can be made from the contents of a bin 20^{ft} long, 10^{ft} wide, and 6^{ft} deep, filled with wheat?

46. How many tons of ice can be packed in a building 60^{ft} long, 40^{ft} wide, and 30^{ft} high?

47. How many tons of coal will a bin 20^{ft} long, 16^{ft} wide, and 12^{ft} deep contain, allowing $40^{cub\ ft}$ to a ton?

48. How many gallons of water will a cistern 8^{ft} deep, and 6^{ft} square at the bottom, hold?

49. A merchant imports 40^{hl} of wine. What must be the dimensions, in feet, of a cubical tank which will just hold it?

50. After a shower, I find a dish, placed in the open air, filled with water to the depth of half an inch. How many tons of water have fallen upon one square mile, the fall being supposed everywhere the same?

51. How many bricks $8^{in} \times 4^{in} \times 2^{in}$ will be required to build a wall 50^{ft} long, 30^{ft} high, and $16\frac{1}{2}^{in}$ thick, allowing $\frac{1}{4}^{in}$ in each dimension for the mortar?

52. How many cords of wood in a pile 60^{ft} long, 8^{ft} wide, and 7^{ft} high?

NOTE.—One *cord* of wood is a pile 8^{ft} long, 4^{ft} wide, and 4^{ft} high.

53. A stick of timber is $14^{in} \times 18^{in}$ at the end. What length of it will contain 30 cubic feet?

54. Find the contents, in *board feet*, of a board 18^{ft} long, 15^{in} wide, and 2^{in} thick?

NOTE.—*One board foot* is 1^{ft} long, 1^{ft} wide, and 1^{in} thick.

55. Find the cost of 20 planks, each 16^{ft} long, 18^{in} wide, and 3^{in} thick, at \$2.00 per hundred feet, board measure.

56. How many feet of lumber, board measure, will be required to make a box $6^{ft} \times 4^{ft} \times 3^{ft}$, outside measurement, the thickness of the lumber to be $1\frac{1}{2}^{in}$?

57. Find the volume of a 3-sided regular prism if a lateral edge = a side of the base = 24^{cm}.

58. In a regular hexagonal prism the height = 3 times the less radius of the base = 60^{cm}. Find the height of a similar prism, made of glass, which weighs 6^{kg}.

59. How high must a regular hexagonal prism be to hold one hectoliter if one side of the base = 64^{cm}?

60. A cylindrical vessel, the area of whose base = 5^{qm}, contains 75^{hl} of water when full; find the height of the vessel.

61. Give the dimensions of a cylindrical vessel which will hold 1000 liters. Is this question determinate or indeterminate? Why?

62. Required the dimensions of a similar cylinder that will hold $\frac{1}{8}$ as much?

63. If a vessel holding one hectoliter is a cylinder whose height is equal to the diameter of the base, find the height.

64. A cylindrical cistern 3.45^{m} deep and 1.1^{m} wide is to be lined with stones, each of which, mortar included, measures 12^{cm} by 6^{cm}. How many stones are required?

65. A cylindrical boiler is 58^{m} long and has a diameter of 0.9^{m}. Find the area of the convex surface exposed to the fire, if this surface intersects the circular end of the boiler in an arc of 130°.

66. If a glass cylinder 0.35^{m} in diameter and 1^{m} in height is filled with water, and the water is then poured into another cylinder 0.8^{m} in diameter, how high will the water rise in the latter vessel?

67. Out of a cylindrical reservoir 8.8^{m} in diameter water flows at the rate of 2^{l} per second. Through what distance will the surface of the water fall in one hour?

68. About the convex surface of a cylinder 0.8^{m} in diameter and 1.2^{m} high, a cord 2^{mm} in diameter is wound until the surface is completely covered. How many meters of cord are required?

69. A well is to be 15^{m} deep and 2.3^{m} in diameter. The price for digging it is to be $0.40 per cubic meter for the first three meters of depth, $0.25 more for the next three meters, and so on, $0.25 being added to the price for each successive three meters of depth. Find the cost of digging the well.

70. A cast-iron cylinder 3^{m} long and 20^{cm} in diameter is reduced in a turning lathe to a diameter of 18^{cm}. Find the loss in weight.

71. The horizontal section of a bath-tub 1^{m} high is an ellipse whose axes are 1.8^{m} and 0.64^{m}. How many liters of water will it hold? (See § **221**.)

72. How large a cylinder can be made by rolling up a rectangular sheet of zinc 80^{cm} by 60^{cm}, if the height of the cylinder is 80^{cm}? How large will the cylinder be if the height is 60^{cm}?

73. An iron cylinder 80^{cm} long and weighing 56^{kg} is bored through the centre until $\frac{3}{4}$ of the iron is removed; find the thickness of what is left.

74. A cylindrical vessel 4^m high and 1.6^m in diameter is half-full of water. Through an orifice at the bottom 4^{cm} in diameter the water is running out at the rate of 6^m a second, and through a pipe 2^{cm} in diameter it is running into the vessel at the rate of 12^m a second; find the level of the water after one-quarter of an hour.

75. What must be the diameter of a supply-pipe, for the vessel in the last exercise, which will supply water as fast as it runs out?

76. What must be its diameter in order that the vessel may be filled in ten minutes?

77. What must be its diameter in order that the vessel may be emptied in one hour?

78. The piston of a pump is 36^{cm} in diameter, and moves through a space of 50^{cm}. How many liters of water are thrown out by 100 strokes?

79. When a body is placed under water, in a cylinder 60^{cm} in diameter, the level of the water is observed to rise 30^{cm} (*Fig. 295*); find the volume of the body.

80. How much will a brass cylinder weigh under water if the height $= 64^{cm}$ and the diameter of the base $= 40^{cm}$? (See Exercise 26.)

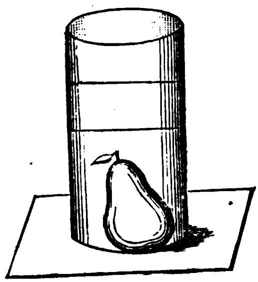

Fig. 295.

§ 276. Pyramids and Cones.

81. In a regular hexagonal pyramid a side of the base $= 1^m$, and the height of the pyramid $= 1.8^m$; find (*i.*) the total surface; (*ii.*) the volume.

82. The roof of a summer-house has the shape of a regular octagonal pyramid; its height $= 15^{ft}$, and a side of its base $= 9^{ft}$. Find the cost of the boards, 1^{in} thick, necessary to make it, at 2 cents a board foot.

83. A vat has the shape of an inverted frustum of a square pyramid; a side of the upper base $= 4^m$, a side of the lower $= 2^m$, and the depth of the vat $= 5^m$. How much water will it hold?

84. The pyramid at Ghiza, near Memphis in Egypt, is about 144^m high; the base is a square whose side $= 187^m$, and the top also a square whose side $= 3.7^m$. If the pyramid were solid, what would be its volume?

85. Divide a pyramid 24^{cm} high into two equal parts by a plane parallel to the base.

86. Find (*i.*) the total surface, (*ii.*) the volume, of a regular tetrahedron whose edge $= 80^{cm}$.

87. The base of a pyramid is a square, and its sides are equilateral triangles; find its volume if a side of the square $= 40^{cm}$.

88. The edge of a regular octahedron $= 2^{m}$; find (*i.*) its surface; (*ii.*) its volume; (*iii.*) the ratio of its volume to that of a cube having the same edge.

89. A hill, having the form of a cone, is to be removed, and the base divided into building-lots. The slant height of the hill $= 120.8^{m}$, and the circumference of the base $= 750^{m}$. (*i.*) How many cubic meters of earth must be removed? (*ii.*) If the base is divided into six equal lots, how large will each be?

90. The height of a cone $= 86^{cm}$, and the slant height is twice as much; find (*i.*) the area of the base; (*ii.*) the volume of the cone.

91. A tower has a conical roof 5.8^{m} high and 12.6^{m} in diameter; find its area.

92. Three leaden cones, each 40^{cm} high, and having for diameters 12^{cm}, 24^{cm}, and 36^{cm}, respectively, are melted together and cast into one having the same height; find its diameter.

93. If the three cones of the last exercise had been cast in the shape of a cube, what would be the edge of the cube.

94. How far from the base must a cone of sugar, 54^{cm} high and 18^{cm} in diameter, be cut in two, parallel to the base, in order that the parts may be equal?

95. Show how the cone of the last exercise may be divided by planes parallel to the base into three equal parts.

96. Out of a circular piece of sheet-tin 86^{cm} in diameter a sector with the angle 150° is cut, and this is then rolled into the shape of a cone; find the volume of this cone.

97. Explain how you would find the height of a cone having given the circumference of the base and the slant height?

98. I wish to find the value of a straight pine tree that tapers to a point, the circumference of the base being 4^{m}. The tree casts a shadow 30^{m} long at the same time that the shadow of a vertical rod 1.2^{m} long is 1.8^{m}. What is the tree worth at $8 per cubic meter (branches, etc., being regarded as worth the price of felling and trimming)?

99. A granite column has the shape of the frustum of a cone; the base is 2.12^{m} in circumference, the top 1.5^{m}, and the height $= 5^{m}$. Find its weight.

100. A vessel holding 12^{l} has the form of the frustum of a cone; the lower base is 28^{cm} in diameter, and the upper base 24^{cm}; find its height.

101. How much sheet tin is required to make a speaking-tube 1.42^{m} long, and 43^{cm} and 36^{cm} in diameter at the ends?

102. How much metal is required to make a tin can, the diameter of the bottom being 36^{cm}, that of the open top 16^{cm}, and the height being 48^{cm}?

103. Assuming the cask (*Fig. 296*) to consist of two frustums of cones placed with their greatest bases together at the middle, find its volume, having given that the diameter of the middle section $= 76^{cm}$, the diameter of each end $= 66^{cm}$, and length $= 1^{m}$. Will the answer obtained be too large or too small?

Fig. 296.

104. A more accurate value for the volume of a cask may be found by use of the formula,

$$\text{Volume} = \tfrac{1}{3}\,\pi\, h\,(r^2 + 2\,R^2),$$

where $h =$ the length, $r =$ radius of one end, $R =$ radius of the greatest section. Find the volume of the preceding cask by means of this formula.

105. How many bottles of wine, each holding 0.6^{l}, can be filled from a cask in which the greatest and the least diameters are 86^{cm} and 62^{cm}, respectively, and the length 1.5^{m}?

106. How many gallons will a cask hold whose length is 3^{ft}, and greatest and least circumferences 12^{ft} and 9^{ft}, respectively?

§ 277. The Sphere.

107. How many bullets 15^{mm} in diameter can be cast from 10^{kg} of lead?

108. The English national debt is about \$4,500,000,000. What would be the diameter of a sphere of gold having the same value, assuming 1^{kg} of gold as worth \$720?

109. The diameter of the sun is about 113 times that of the earth. Find the volume of the sun, assuming the diameter of the earth to be 8000 miles.

110. Eighty bullets, equal in size, are thrown into a cylindrical vessel 68^{cm} in diameter containing water. If the level of the water rises 2^{cm}, find the diameter of a bullet.

111. A body sinks in water till the weight of the water displaced = weight of the body. How heavy is a sphere 0.2^{m} in diameter which floats half under water?

112. Find the weight of a wooden sphere 0.36^{m} in diameter if one-sixth of its bulk is below the water-level when it floats on water.

113. If a wooden sphere 20^{cm} in diameter is reduced to one-half its original volume in a lathe, what will then be its diameter?

114. Find the diameter of a sphere made by melting together three spheres whose diameters are 1.2^m, 0.8^m, and 0.4^m respectively.

115. A sphere 0.6^m in diameter is reduced in size by turning till its diameter is one-twelfth less than at first. How much is the volume reduced?

116. The outer circumference of a hollow sphere $= 1.4^m$, the thickness $= 24^{mm}$. Find the interior volume.

117. An arch in the form of a hemisphere has an inner diameter of 4.8^m, and its thickness $= 0.7^m$. Find how many cubic meters of stone it contains.

118. What is the value of a hemispherical copper kettle 1^m inside diameter, if the thickness of the copper $= 1^{cm}$, and copper is worth $\$0.75$ per kilogram?

119. The diameter of a balloon $= 30^m$; the material of which it is made weighs 0.5^{kg} per square meter; and the car attached below weighs 40^{kg}. Find its ascensional force if air weighs 1.3^g per liter, and the balloon is filled with coal gas which is half as heavy as air.

Hint. — The ascensional force = weight of the air displaced by the balloon — weight of the balloon, the car, and the gas in the balloon.

120. What must be the outer diameter of a hollow sphere 25^{mm} thick, in order that it may hold just 60 liters?

121. How many bodies as large as the moon could be made from the earth, taking the diameter of the latter body as four times that of the former?

122. If a drop of soap and water 4^{cm} in diameter is blown into a bubble 15^{cm} in diameter, find the thickness of the bubble.

123. The two frigid zones contain 0.082 of the earth's surface. What is the height of each zone? What is the diameter of the Arctic circle?

§ 278. Equivalent Surfaces and Solids.

124. A cube and a sphere have the same volume; which has the greater surface?

125. A cube and a sphere have the same surface; which has the greater volume?

The radius of the base of a cone $= 10^{cm}$, the height of the cone $= 20^{cm}$. Find, —

126. The edge of an equivalent cube.

127. The height of an equivalent cylinder, etc., having the same base.

128. The radius of an equivalent sphere.

129. How high must a four-sided regular prism be, whose base is $16^{cm} \times 12^{cm}$, that it may be equivalent to a cube whose edge $= 10^{cm}$?

130. A cube whose edge $= 10^{cm}$ is cut into two equal parts by a plane parallel to the base, and the parts are then placed with the half sides together so as to form a prism. Compare this prism and the cube as regards (*i.*) volumes; (*ii.*) surface; (*iii.*) bases; (*iv.*) heights.

Note 1.—Equivalent prisms have not always equivalent surfaces.

Note 2.—In equivalent prisms (also cylinders, pyramids, and cones), the bases are ***inversely proportional*** to the heights (that is, the *less* the base the ***greater*** the height, and conversely, so that the product of the two factors, base and height, shall be always equal).

131. In two equivalent cylinders the ratio of the bases is 3 : 5. What is the ratio of the heights?

132. The dimensions of a cylinder are: diameter of the base $= 7^{cm}$, height $= 12^{cm}$. Find the height of an equivalent cylinder, the diameter of its base being 6^{cm}.

133. Compare the surfaces of the two bodies in Exercise 129. Which is the greater, and by how much?

Note.—Among all equivalent four-sided regular prisms the cube has the least surface.

134. A cube with an edge of 8^{cm} has the same surface as a four-sided regular prism whose base is $6^{cm} \times 4^{cm}$; compare their volumes. Which is the greater, and by how much?

Note.—Among all four-sided regular prisms with equivalent surfaces the cube has the greatest volume.

135. The base of a square pyramid is $6^{cm} \times 6^{cm}$, and the height $= 12^{cm}$; find the height of an equivalent prism with an equal base.

136. Transform a cylinder 10^{cm} in diameter and 12^{cm} high into an equivalent square pyramid with a base $8^{cm} \times 8^{cm}$.

137. Transform a cube with an edge of 8^{cm} into an equivalent cylinder the diameter of whose base shall be 10^{cm}; then compare the surfaces of the two bodies.

138. Transform the above cube into a cylinder with an equivalent surface, the diameter of its base to be 10^{cm}; then compare the volumes of the two bodies.

139. Transform a cylinder, in which the diameter of the base and the height are each equal to 10^{cm}, into an equivalent cylinder having a height of 16^{cm}; then compare the surfaces of the two cylinders.

Note.—Among equivalent cylinders that has the least surface in which the diameter of the base and the height are equal.

140. Slant height of a cone = 8^{cm}, and diameter of its base = 6^{cm}; find the height of a cylinder 4^{cm} in diameter, having an equivalent convex surface.

141. Transform a sphere 6^{cm} in diameter into an equivalent cylinder, the diameter of its base to be the same as that of the sphere; then compare their surfaces.

142. A leaden sphere 9^{cm} in diameter is recast in the shape of a cube; find the edge of the cube.

143. Another sphere of the same size is recast in the form of a cylinder 5^{cm} in diameter; find the height of the cylinder.

144. What must be the height of a cylinder whose base is 4^{cm} in diameter if it has the same surface as that of a sphere 4^{cm} in diameter?

145. A sphere is 10^{cm} in diameter, and the base of an equivalent cylinder has the same diameter; compare the surfaces of the two bodies.

Note.—Among all equivalent bodies the sphere has the least surface.

146. A sphere and a cylinder have equivalent surfaces; the sphere is 10^{cm} in diameter, and so likewise is the base of the cylinder; compare the volumes of the two bodies.

Note.—Among all bodies having equivalent surfaces the sphere has the greatest volume.

§ 279. Ratios of Surfaces and of Solids.

147. Two prisms have bases each containing 36^{qm}; what is the ratio of their volumes if their heights are 10^{cm} and 15^{cm}?

148. Compare in general the volumes of two prisms if they have equivalent bases.

149. Compare in like manner the volumes of two cylinders, two pyramids, and two cones, with equivalent bases.

150. The volumes of two prisms with equivalent bases are 360^{cbm} and 560^{cbm}; what is the ratio of their heights? If the height of one is 27^{cm}, find that of the other.

151. A cone 8^{cm} high is reduced in size till the height is 3.5^{cm}, the base remaining the same. What fractional part of the whole is left?

152. Into the ends of a cylinder 12^{cm} long two cone-shaped holes 3^{cm} deep are bored. If the cylinder weighed at first 600^{g}, what will it weigh after the holes are made?

153. If a cylinder and a cone have equal bases, what must be the ratio of their heights in order that their volumes may be equal?

154. The heights of two prisms are each 20^{cm}; compare their volumes if the base of one is four times that of the other.

155. Compare in general the volumes of two prisms, two cylinders, two pyramids, or two cones, having equal heights.

156. A cone 6^{cm} in diameter is turned in a lathe until the diameter is reduced to 5^{cm}, the height remaining unchanged; what part of the whole cone is left?

157. If the above is turned till it is reduced to one-fourth its original volume, what will be its diameter?

The edge of a cube $= 10^{cm}$; find, —

158. Volume of the largest cylinder that can be made from it.

159. Volume of the largest sphere.

160. Volume of the largest cone.

161. Compare the volumes of the above cube, cylinder, sphere, and cone.

The diameter of a cylinder = its height $= 10^{cm}$; find, —

162. Volume of the largest sphere that can be turned from it;

163. Volume of the largest cone.

164. Compare the volumes of the above cylinder, sphere, and cone.

165. What part of a hemisphere is the largest cone that can be made from it?

166. If the diameter and the height of a cylinder are equal, compare the convex surface with the areas of the bases.

167. If the diameter and the height of a cone are equal, compare the convex surface with the area of the base.

168. If the convex surface of a cylinder = sum of areas of the bases, compare the height with the radius of the base.

169. I want a cylinder whose convex surface shall be six times the area of the base. Find the ratio of the height to the radius of the base.

170. What relation must exist between the height of a cone and its diameter in order that the convex surface may be equal in area to the base?

171. Compare the convex surfaces of a cylinder and a cone having equal bases and heights. Compare, also, their total surfaces.

172. Compare the surfaces of two spheres whose diameters are 6^{cm} and 7^{cm}. What, in general, is the relation between the surfaces of spheres differing in diameter?

173. A cylinder and a hemisphere have equal bases and heights. Compare (*i.*) their convex surfaces; (*ii.*) their total surfaces; (*iii.*) their volumes.

174. The diameter of a cylinder = its height $= 10^{cm}$; find the volume of the largest rectangular parallelopiped that can be made from it.

175. Compare the volumes of a sphere and the largest cube that can be made from it.

REVIEW OF CHAPTER XIII.

I.—Surfaces of Bodies.

§ **253**. Surface of a prism.
§ **254**. Surface of a cylinder.
§ **255**. Surface of a pyramid.
§ **256**. Surface of a cone.
§ **257**. Surface of the frustum of a cone.
§ **258**. Surface of a sphere.

II.—Volumes of Bodies.

§ **259**. Definition of Volume. Units of volume.
§ **260**. Measurement of volumes. Analysis of the different cases to be considered.
§ **261**. Volume of the rectangular parallelopiped.
§ **262**. Volume of the right parallelopiped.
§ **263**. Volume of the oblique parallopiped.
§ **264**. Volume of the triangular prism.
§ **265**. Volume of any prism.
§ **266**. Volume of the cylinder.
§ **267**. Volume of the triangular pyramid.
§ **268**. Volume of any pyramid.
§ **269**. Volume of the cone.
§ **270**. Results of §§ **261–269** summed up in two rules.
§ **271**. Volume of a sphere.
§ **272**. Relation between volume and weight.
§ **273**. Equivalent solids, similar solids, equal solids.
§ **274**. Two laws applicable to similar solids.

III.—Exercises and Applications.

§ **275**. Prisms and cylinders.
§ **276**. Pyramids and cones.
§ **277**. The sphere.
§ **278**. Equivalent surfaces and solids.
§ **279**. Ratios of surfaces and solids.